Tutorial on Hardy fields and transseries

Part I: Hardy fields

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Overview

Hardy fields and *transseries* are complementary approaches to a "tame" part of analysis. In these lectures I plan to give an introduction to these two topics assuming no knowledge of either.

> *Hardy fields are the natural domain of asymptotic analysis, where all rules hold, without qualifying conditions.*
>
> —Maxwell Rosenlicht

They are one-dimensional relatives of o-minimal structures, and as such they occupy a central role in the intersection of analysis, model theory, and dynamical systems. But they have also found applications in various other parts of mathematics, notably in ergodic theory.

Only introduced in the 1980s, transseries are formal objects which allow us to model the asymptotic behavior of elements in Hardy fields, and are often are easier to handle (no convergence considerations, etc.). They arose independently in analysis (Écalle: Dulac's Problem) and logic (Dahn-Göring: Tarski's Problem).

Overview

My talks will go hand in hand with those of Tobias Kaiser, who will focus on the connections between Hardy fields, o-minimality, and Hilbert's 16th Problem, whereas I will concentrate on model theoretic and algebraic aspects (but some baby analysis will be involved).

Plan for my lectures

- I Hardy fields
- II Transseries
- III Asymptotic differential algebra
- IV Model theory of transseries
- V Maximal Hardy fields

I. Hardy fields

For $a \in \mathbb{R}$ let

$$\mathcal{C}_a \;:=\; \text{ring of continuous functions } [a, +\infty) \to \mathbb{R}.$$

For $f \in \mathcal{C}_a$ ($a \in \mathbb{R}$) and $g \in \mathcal{C}_b$ ($b \in \mathbb{R}$) we say that

f **and** g **have the same germ at** $+\infty$ $:\Leftrightarrow$ $f(t) = g(t)$ for all $t \gg 0$.

This defines an equivalence relation on the set $\bigcup_a \mathcal{C}_a$ (disjoint union). The equivalence class of $f \in \mathcal{C}_a$ is the **germ of** f at $+\infty$.

$$\mathcal{C} \;:=\; \text{ring of germs at } +\infty \text{ of functions } f \in \mathcal{C}_a \ (a \in \mathbb{R}).$$

Given a property P of real numbers and $f \in \mathcal{C}$ we say that $P\big(f(t)\big)$ **holds eventually** if $P\big(f(t)\big)$ holds for all $t \gg 0$. Thus

$$f = 0 \quad\Longleftrightarrow\quad f(t) = 0 \text{ eventually.}$$

We have a partial ordering on $\mathcal{C}$ given by

$$f \leqslant g \quad :\Longleftrightarrow \quad f(t) \leqslant g(t), \text{ eventually.}$$

Naturally we have $\mathbb{R} \subseteq \mathcal{C}$ (as an ordered subfield).

We also define

$$\begin{aligned} f < g \quad &:\Longleftrightarrow \quad f \leqslant g \;\&\; f \neq g, \\ f <_{\mathrm{e}} g \quad &:\Longleftrightarrow \quad f(t) < g(t), \text{ eventually,} \\ &(\;\Longrightarrow \quad f < g.) \end{aligned}$$

Examples (units of the ring $\mathcal{C}$)

$$f \in \mathcal{C}^{\times} \quad \Longleftrightarrow \quad f(t) \neq 0, \text{ eventually} \quad \Longleftrightarrow \quad f <_{\mathrm{e}} 0 \text{ or } 0 <_{\mathrm{e}} f.$$

Thus the germ at $+\infty$ of $\sin x$ is $\neq 0$ in $\mathcal{C}$ but not a unit of $\mathcal{C}$.

Asymptotic relations on $\mathcal{C}$

$f \preccurlyeq g \quad :\Longleftrightarrow \quad |f| \leqslant c|g|$ for <u>some</u> $c \in \mathbb{R}^{>}$ — g dominates f

$f \prec g \quad :\Longleftrightarrow \quad g \in \mathcal{C}^{\times}$ and $|f| \leqslant c|g|$ for <u>all</u> $c \in \mathbb{R}^{>}$ — g strictly dominates f

$f \asymp g \quad :\Longleftrightarrow \quad f \preccurlyeq g$ and $g \preccurlyeq f$ — f and g are asymptotic

$f \sim g \quad :\Longleftrightarrow \quad f - g \prec g$ — f and g are equivalent.

(Indeed, $\asymp$ and $\sim$ are equivalence relations on $\mathcal{C}$ respectively $\mathcal{C}^{\times}$.)

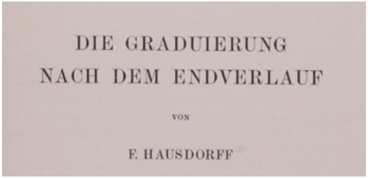
Examples (with $x =$ germ at $+\infty$ of the identity function)

- $1 \prec x \prec x^2 \prec \cdots \prec x^n \prec x^{n+1} \prec \cdots$, so

$$a_0 + a_1 x + \cdots + a_n x^n \sim a_n x^n \quad \text{for } a_0, \ldots, a_n \in \mathbb{R},\ a_n \neq 0.$$

- $x^2 \cdot \sin x \not\preccurlyeq x, \quad x \not\preccurlyeq x^2 \cdot \sin x.$

Hausdorff fields


DIE GRADUIERUNG
NACH DEM ENDVERLAUF
VON
F. HAUSDORFF

Definition

A subfield of $\mathcal{C}$ is called a **Hausdorff field**.

Example

$\mathbb{R}[x]^{\neq} := \mathbb{R}[x] \setminus \{0\} \subseteq \mathcal{C}^{\times} \implies \mathbb{R}(x) \subseteq \mathcal{C}$ is a Hausdorff field.

Let H be a Hausdorff field and $f, g \in H$. Then

$$f \neq 0 \quad \implies \quad f \in H^{\times} \subseteq \mathcal{C}^{\times} \quad \implies \quad f >_{\mathrm{e}} 0 \text{ or } f <_{\mathrm{e}} 0,$$

so $\leqslant$ restricts to a *total* ordering on H making H an ordered field.

As a consequence

$$f \not\preccurlyeq 1 \quad \implies \quad |f| > n \text{ for each } n \quad \implies \quad 1 \prec f,$$

so unlike for arbitrary germs, one of $f \preccurlyeq g$ or $g \preccurlyeq f$ always holds.

Hence $\preccurlyeq$ restricts to a dominance relation on H:

Definition (for a field K)

A **dominance relation** on K is a binary relation $\preccurlyeq$ on K such that

- (D0) $1 \not\preccurlyeq 0$;
- (D1) $f \preccurlyeq f$;
- (D2) $f \preccurlyeq g$ and $g \preccurlyeq h \Rightarrow f \preccurlyeq h$;
- (D3) $f \preccurlyeq g$ or $g \preccurlyeq f$;
- (D4) $f \preccurlyeq g \Leftrightarrow fh \preccurlyeq gh$, provided $h \neq 0$;
- (D5) $f \preccurlyeq h$ and $g \preccurlyeq h \Rightarrow f + g \preccurlyeq h$.

Let $\preccurlyeq$ be a dominance relation on K. We have the subring

$$\mathcal{O} := \{f \in K : f \preccurlyeq 1\}$$

of K, which is a **valuation ring** of K: $\boxed{f \in K \setminus \mathcal{O} \ \Rightarrow \ 1/f \in \mathcal{O}}$

Each valuation ring of K arises from a unique dominance relation.

Dominance relations

Let K be a **valued field**: a field with a dominance relation on it. Put

$$f \prec g \;:\Longleftrightarrow\; f \preccurlyeq g \,\&\, g \not\preccurlyeq f, \qquad f \asymp g \;:\Longleftrightarrow\; f \preccurlyeq g \,\&\, g \preccurlyeq f.$$

The valuation ring $\mathcal{O} = \mathcal{O}_K$ has a unique maximal ideal, namely

$$\mathcal{o} = \mathcal{o}_K := \{f \in K : f \prec 1\}.$$

For $f \in K$ let $\quad vf := \left(\begin{array}{l}\text{equivalence class of } f \text{ with respect}\\ \text{to the equivalence relation } \asymp \text{ on } K\end{array}\right).$

Two subordinate structures

1. The **residue field** $\boxed{\boldsymbol{k} = \boldsymbol{k}_K := \mathcal{O}/\mathcal{o}}$ of K.

   The map $f \mapsto f + \mathcal{o} \colon \mathcal{O} \to \boldsymbol{k}$ is the **residue morphism**.

2. The (ordered) **value group** $\boxed{\Gamma = \Gamma_K := \{vf : f \in K^\times\}}$ of K, with group operation $+$ and ordering $\leqslant$ given by

$$vf + vg = v(f \cdot g), \qquad vf \geqslant vg \;:\Longleftrightarrow\; f \preccurlyeq g.$$

   The map $v \colon K^\times \to \Gamma$ is the **valuation** of K.

The dominance relation on a Hausdorff field

Let H be a Hausdorff field, turned into a valued field by the restriction of the relation $\preccurlyeq$. Then the restrictions of $\prec$, $\asymp$ from $\mathcal{C}$ to H give exactly the relations on H from the previous slide. Also,

$$\mathcal{O} = \big\{ f \in H : |f| \leqslant n \text{ for some } n \big\},$$
$$\smallo = \big\{ f \in H : |f| \leqslant 1/n \text{ for all } n \geqslant 1 \big\}.$$

Equip $\boldsymbol{k} = \mathcal{O}/\smallo$ with the unique ordering making it an ordered field and the residue morphism $\mathcal{O} \to \boldsymbol{k}$ order-preserving. Then there is a unique embedding $\boldsymbol{k} \to \mathbb{R}$, which is onto if $\mathbb{R} \subseteq H$.

Example: $H = \mathbb{R}(x)$

Then $\Gamma = \mathbb{Z}v(x)$ with $v(x) < 0 = v(1)$, and

$$v(p/q) = (\deg p - \deg q)v(x) \qquad \text{for } p, q \in \mathbb{R}[x]^{\neq}.$$

Algebraic extensions of Hausdorff fields

An ordered field F is real closed if $F[i]$ ($i^2 = -1$) is algebraically closed. Equivalently (Artin-Schreier):

(R1) every $f \in F^>$ has a square root in F, and

(R2) each odd-degree $P \in F[Y]$ has a zero in F.

Theorem (essentially Hausdorff)

$$H^{\mathrm{rc}} := \{ y \in \mathcal{C} : P(y) = 0 \text{ for some } P \in H[Y]^{\neq} \}$$

is a real closed Hausdorff field extending H.

Key part of the proof:

if $y \in H^{\mathrm{rc}}$, then $P(y) = 0$ for some monic irreducible $P \in H[Y]$.

To see this let

$$P \;=\; Y^d + P_1 Y^{d-1} + \cdots + P_d \in H[Y] \qquad (P_1, \ldots, P_d \in H)$$

be any monic polynomial in $H[Y]$ of degree $d \geqslant 1$.

Take $a \in \mathbb{R}$ and representatives of the P_j in $\mathcal{C}_a$. This yields for $t \geqslant a$:

$$P(t,Y) \ := \ Y^d + P_1(t)Y^{d-1} + \cdots + P_d(t) \in \mathbb{R}[Y].$$

Lemma 1 (parametrizing the real zeros of $P(t,Y)$)

Suppose P is irreducible. Then there are $y_1 <_{\mathrm{e}} \cdots <_{\mathrm{e}} y_m$ in $\mathcal{C}$ such that the distinct real zeros of $P(t,Y)$ are $y_1(t), \ldots, y_m(t)$, eventually.

Proof.

Take $A, B \in H[Y]$ with $1 = AP + BP'$. Then

$$1 \ = \ A(t,Y)P(t,Y) + B(t,Y)P(t,Y)', \qquad \text{eventually.}$$

Hence $P(t,Y)$ has exactly d distinct complex zeros, eventually.
Now use “continuity of roots”. □

Similarly one shows:

Lemma 2

Let $P \neq Q$ in $H[Y]$ be monic and irreducible. Then for all $y, z \in \mathcal{C}$ with $P(y) = Q(z) = 0$ we either have $y <_{\mathrm{e}} z$ or $y >_{\mathrm{e}} z$.

Suppose now $y \in H^{\mathrm{rc}}$ with $P(y) = 0$. Write

$$P = Q_1^{e_1} \cdots Q_n^{e_n} \quad (e_j \geqslant 1, Q_j \in H[Y] \text{ distinct, monic irreducible}).$$

Lemmas 1 and 2 yield $y_1, \ldots, y_m \in \mathcal{C}$ such that

1. eventually, $y_1(t) < \cdots < y_m(t)$ are the real zeros of the $Q_1(t, Y), \ldots, Q_n(t, Y) \in \mathbb{R}[Y]$ (thus of $P(t, Y)$);
2. for each $i \in \{1, \ldots, m\}$ there is a unique $j \in \{1, \ldots, n\}$ with $Q_j\big(t, y_i(t)\big) = 0$, eventually.

Continuity and the connectedness of halflines $[a, +\infty)$ yields a single i with $y_i = y$, and thus $Q_j(y) = 0$ for some j. □

Composition

Let $g \in \mathcal{C}$ be eventually strictly increasing such that $g > \mathbb{R}$, with compositional inverse $g^{\text{inv}} \in \mathcal{C}$. The composition operation

$$f \mapsto f \circ g \, : \, \mathcal{C} \to \mathcal{C}, \qquad (f \circ g)(t) \; := \; f\big(g(t)\big) \text{ eventually,}$$

is an $\mathbb{R}$-algebra automorphism of $\mathcal{C}$, with inverse $f \mapsto f \circ g^{\text{inv}}$, which maps H isomorphically onto the Hausdorff field $H \circ g$.

Example

Let $H = \mathbb{R}(x)$ and $g = x + \sin x$; then $H \circ g = \mathbb{R}(x + \sin x)$.

We say that H is **closed under composition** if for all eventually strictly increasing $g \in H$ with $g > \mathbb{R}$, we have $H \circ g \subseteq H$; similarly we define when H is **closed under inverses**.

Example

$H = \mathbb{R}(x)$ is closed under composition but not under inverses.

Let's now bring differentiation into the picture: for $r = 0, 1, 2, \ldots$

$$\mathcal{C}^r \;:=\; \left\{ \begin{array}{l} \text{ring of germs } f \in \mathcal{C} \text{ having an } r\text{-times continuously} \\ \text{differentiable representative } [a, +\infty) \to \mathbb{R} \ (a \in \mathbb{R}), \end{array} \right.$$

and $\mathcal{C}^{<\infty} \;:=\; \bigcap_r \mathcal{C}^r$, a differential ring

(with differential subrings $\mathcal{C}^\infty$ and $\mathcal{C}^\omega$).

Definition (Bourbaki)

A **Hardy field** is a differential subfield of $\mathcal{C}^{<\infty}$.

Analogously one defines $\mathcal{C}^\infty$**-Hardy fields** or $\mathcal{C}^\omega$**-Hardy fields:**

$$\{\mathcal{C}^\omega\text{-Hardy fields}\} \;\subseteq\; \{\mathcal{C}^\infty\text{-Hardy fields}\} \;\subseteq\; \{\text{Hardy fields}\}$$

All these inclusions are proper, but this is not obvious.

Most Hardy fields that occur "in nature" are analytic. Easy examples:

$$\mathbb{Q} \;\subseteq\; \mathbb{R} \;\subseteq\; \mathbb{R}(x) \;\subseteq\; \mathbb{R}(x, \mathrm{e}^x) \;\subseteq\; \mathbb{R}(\log x, x, \mathrm{e}^x)$$

Let H be a Hardy field. Then H is a Hausdorff field. We view H as an ordered valued field as explained before. Note:

$$f \in H \quad \Longrightarrow \quad f' \in H \quad \Longrightarrow \quad \operatorname{sign} f'(t) \text{ eventually constant,}$$

so f is eventually monotonic, hence $\lim_{t \to +\infty} f(t) \in \mathbb{R} \cup \{\pm\infty\}$ exists.

We have

$$f \preccurlyeq g \iff |f| \leqslant c|g| \text{ for some } c \in \mathbb{R}^{>} \iff \lim_{t \to +\infty} \frac{f(t)}{g(t)} \in \mathbb{R},$$

$$f \prec g \iff |f| \leqslant c|g| \text{ for each } c \in \mathbb{R}^{>} \iff \lim_{t \to +\infty} \frac{f(t)}{g(t)} = 0.$$

Example (for what Rosenlicht meant)

Suppose $0 \neq f, g \not\asymp 1$ are in a Hardy field. Then (l'Hôpital's Rule):

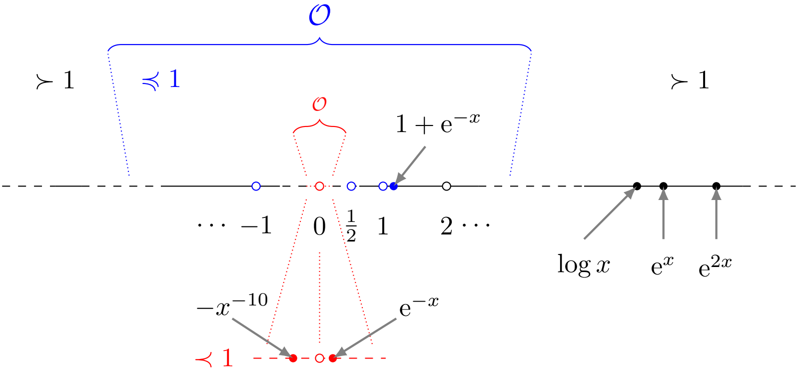
$$f \preccurlyeq g \quad \iff \quad f' \preccurlyeq g'$$

Let $H = \mathbb{R}(\log x, x, \mathrm{e}^x)$. Below is a depiction of the valuation ring

$$\mathcal{O} = \{h \in H : h \preccurlyeq 1\}$$

of H with its maximal ideal of "infinitesimals":

$$\mathcal{o} = \{h \in H : h \prec 1\}.$$



Composition in Hardy fields

Let H be a Hardy field. In order to obtain another Hardy field, not just a Hausdorff field, via composition, requires some care:

Let $\ell \in \mathcal{C}^1$ with $\boxed{\phi := \ell' \in H, \ell > \mathbb{R}.}$

Then $\phi > 0$, and the $\mathbb{R}$-algebra automorphism

$$h \mapsto h^\circ := h \circ \ell^{\text{inv}}$$

of $\mathcal{C}^{<\infty}$ maps H onto the Hardy field $H^\circ = H \circ \ell^{\text{inv}}$:

$$(h^\circ)' = (\phi^{-1} h')^\circ.$$

NB: the ordered field isomorphism $h \mapsto h^\circ \colon H \to H^\circ$ is not a *differential* field isomorphism!

Let H be a Hardy field.

Theorem (A. Robinson)

H^{rc} is a Hardy field.

For this let $P_0, \dots, P_d \colon [a, +\infty) \to \mathbb{R}$ be C^1. For $t \geqslant a$ and $y \in \mathbb{R}$:

$$P(t,y) := P_0(t) + P_1(t)y + \cdots + P_d(t)y^d,$$
$$P'(t,y) := P_1(t) + 2P_2(t)y + \cdots + dP_d(t)y^{d-1}, \text{ and}$$
$$P^\partial(t,y) := P_0'(t) + P_1'(t)y + \cdots + P_d'(t)y^d.$$

Let $y \in \mathcal{C}_a$ satisfy $P\big(t, y(t)\big) = 0$ and $P'\big(t, y(t)\big) \neq 0$ for all $t \geqslant a$; then y is C^1 with

$$y'(t) = -P^\partial\big(t, y(t)\big) / P'\big(t, y(t)\big) \qquad \text{for } t \geqslant a.$$

This follows from the Implicit Function Theorem and the Chain Rule.

Hence if $y \in H^{\mathrm{rc}}$, then $y \in \mathcal{C}^1$ and $y' \in H[y] \subseteq H^{\mathrm{rc}}$. □

Next we turn to simple first-order algebraic differential equations.

Theorem (Marić, Singer, 1970s)

Let $F, G \in H[Y]$ *and* $y \in \mathcal{C}^1$ *with*

$$y'G(y) = F(y) \quad \text{and} \quad G(y) \in \mathcal{C}^\times.$$

Then $H[y]$ *is an integral domain and has fraction field* $H(y) \subseteq \mathcal{C}^{<\infty}$*; moreover,* $H(y)$ *is a Hardy field.*

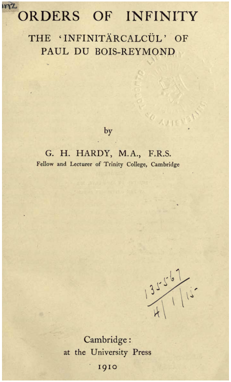
NB: we are not claiming that a solution $y \in \mathcal{C}^1$ of the differential equation $y'G(y) = F(y)$ always exists! (Think $y' = 1 + y^2$.)

Corollary (Hardy, Bourbaki)

$H(\mathbb{R})$ *and* $H(x)$ *are* Hardy *fields, and for* $h \in H$, *so are*

$$H(\textstyle\int h), \quad H(\mathrm{e}^h), \quad H(\log h) \text{ when } h > 0.$$

Hardy’s dream


ORDERS OF INFINITY

THE ‘INFINITÄRCALCÜL’ OF
PAUL DU BOIS-REYMOND

by

G. H. HARDY, M.A., F.R.S.
Fellow and Lecturer of Trinity College, Cambridge

Cambridge:
at the University Press
1910

Hardy defined the field H_{LE} of (germs of) **logarithmic-exponential functions**:

> *the smallest real closed Hardy field containing* $\mathbb{R}(x)$ *which is closed under* $\exp$ *and* $\log$.

Examples of germs in H_{LE}:

$$x^{\sqrt{2}} + 5x - 3x^{-1} \qquad \mathrm{e}^{\mathrm{e}^x} + x^2$$

$$\sinh x = \tfrac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}) \qquad \log\left(\frac{x+1}{x-1}\right)$$

He made a rather audacious claim:

> exponential scales. No function has yet presented itself in analysis the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms.

It turns out that H_{LE} is closed under composition.

But $(\log x \log\log x)^{\mathrm{inv}}$ is not an LE-function. (Liouville, 1830s)

It is not even *asymptotic* to any $h \in H_{\mathrm{LE}}$.
(van den Dries-Macintyre-Marker, van der Hoeven, 1997)

Moreover, H_{LE} is not closed under $\int$. (E.g., $\int \mathrm{e}^{x^2} \notin H_{\mathrm{LE}}$). Thus:

> H_{LE} lacks many closure properties that would make it useful for a comprehensive theory of "tame" asymptotic analysis.

We may modify the definition: for example,

$$\mathrm{Li}(\mathbb{R}) := \begin{cases} \textit{the smallest real closed Hardy field} \\ \textit{which is closed under } \exp \textit{ and } \int. \end{cases}$$

Note: $\mathrm{Li}(\mathbb{R})$ (the **Hardy-Liouville closure** of $\mathbb{R}$) contains x and is closed under $\log$: $(\log h)' = h'/h$ for $h > 0$. (So $H_{\mathrm{LE}} \subseteq \mathrm{Li}(\mathbb{R})$.)

How to go beyond order 1 differential equations?

A cautionary example (Boshernitzan, 1986)

Any $y \in \mathcal{C}^2$ satisfying

$$y'' + y = \mathrm{e}^{x^2}$$

is hardian, i.e., contained in a Hardy field.

(But no two distinct solutions to this equation are in a common Hardy field, and none of them is in $\mathrm{Li}(\mathbb{R})$.)

The growth of germs in $\mathrm{Li}(\mathbb{R})$ is also quite restricted:

Exponential boundedness: $\mathrm{e}_0 = x < \mathrm{e}_1 = \mathrm{e}^x < \mathrm{e}_2 = \mathrm{e}^{\mathrm{e}^x} < \cdots$

For any $h \in \mathrm{Li}(\mathbb{R})$ there is some n such that $h \leqslant \mathrm{e}_n$.

Sjödin (1970) constructed a hardian germ $\mathrm{e}_\omega \in \mathcal{C}^\infty$ such that $\mathrm{e}_\omega \geqslant \mathrm{e}_n$ for each n. (Such e_ω is necessarily differentially transcendental.)

Boshernitzan (1984) showed that one can even take e_ω to be analytic, namely as an analytic solution to the functional equation

$$\mathrm{e}_\omega \circ (x+1) = \exp \circ\, \mathrm{e}_\omega,$$

which was shown to exist by H. Kneser (1940s).

Indeed, every Hardy field extends to an Hardy field H which is *unbounded*: there is no $\phi \in \mathcal{C}$ with $h \leqslant \phi$ for each $h \in H$.

Nevertheless, *G. H. Hardy’s dream of an* ***all-inclusive, maximally stable algebra of “totally formalizable functions”*** (J. Écalle) persists.

In the next lectures we will see partial realizations of this vision.

P. du Bois-Reymond (1831–1889)

F. Hausdorff (1868–1942)

G. H. Hardy (1877–1947)

Tutorial on Hardy fields and transseries

Part II: Transseries

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Plan for my lectures

- I Hardy fields
- II Transseries
- III Asymptotic differential algebra
- IV Model theory of transseries
- V Maximal Hardy fields

In my previous lecture, I introduced Hardy fields: *differential fields of germs of real-valued one-variable functions.*

We met the Hardy field H_{LE} of LE-functions and the larger $\mathrm{Li}(\mathbb{R})$.

Unfortunately, both are quite small: H_{LE} is not closed under $\int$, and $\mathrm{Li}(\mathbb{R})$ is not closed under solving 2nd order linear DEs.

Today we will see a kind of workaround via formal series expansions.

Reminder on Laurent series

The field $\mathbb{R}(\!(x^{-1})\!)$ of (formal) Laurent series over $\mathbb{R}$ in *descending* powers of x consists of all series

$$f(x) = \underbrace{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x}_{\text{infinite part of } f} + a_0 + \underbrace{a_{-1}x^{-1} + a_{-2}x^{-2} + \cdots}_{\text{infinitesimal part of } f}$$

We equip $\mathbb{R}(\!(x^{-1})\!)$ with the *ordering* where $x > \mathbb{R}$, *dominance relation* with $f \preccurlyeq 1 \Leftrightarrow f$ has infinite part 0, and *derivation* $\frac{d}{dx}$.

We naturally have $\mathbb{R}(\!(x^{-1/m})\!) \subseteq \mathbb{R}(\!(x^{-1/mn})\!)$ for $m, n \geqslant 1$, resulting in the ordered valued differential field

$$\mathrm{P}(\mathbb{R}) := \bigcup_{n \geqslant 1} \mathbb{R}(\!(x^{-1/n})\!)$$

of Puiseux series over $\mathbb{R}$.

The ordered field $\mathrm{P}(\mathbb{R})$ turns out to be real closed. (Newton)

A consequence: the elements of the Hardy subfield

$$\mathbb{R}(x)^{\mathrm{rc}} = \{y \in \mathcal{C} : P(y) = 0 \text{ for some } P \in \mathbb{R}(x)[Y]^{\neq}\}$$

of H_{LE} admit an asymptotic expansion at $+\infty$ using Puiseux series:

There is an embedding

$$\mathbb{R}(x)^{\mathrm{rc}} \hookrightarrow \mathrm{P}(\mathbb{R})$$

of ordered valued differential fields.

This embedding cannot be extended to an embedding $H_{\mathrm{LE}} \hookrightarrow \mathrm{P}(\mathbb{R})$:

$\exp x$ *and* $\log x$ *do not make sense in* $\mathrm{P}(\mathbb{R})$.

Question

Can we enlarge $\mathrm{P}(\mathbb{R})$ in a natural way to an ordered differential field of formal series which embeds many more Hardy fields (like $\mathrm{Li}(\mathbb{R})$)?

The general idea:

> *start with* $\mathbb{R}(x)$ *and iteratively close off under* $\exp$, $\log$, *and* *infinite summation*.

In this lecture we will see how to make this idea precise and extend $\mathrm{P}(\mathbb{R})$ to the ordered differential field $\boxed{\mathbb{T}}$ of transseries:

$$\mathrm{e}^{\mathrm{e}^{x}+\mathrm{e}^{x/2}+\mathrm{e}^{x/4}+\cdots}-3\,\mathrm{e}^{x^2}+5x^{\sqrt{2}}-(\log x)^{\pi}+42+x^{-1}+x^{-2}+\cdots+\mathrm{e}^{-x}\,.$$

This ordered differential field turns out to be closed under $\int$ and under solving inhomogeneous order 2 linear differential equations.

This will give us hope that something similar can also be achieved on the Hardy field side.

II. Transseries

Let $(\mathfrak{M}, \preccurlyeq)$ be a linearly ordered set (of *monomials*). Call $\mathfrak{S} \subseteq \mathfrak{M}$ **well-based** if there is no sequence

$$\mathfrak{m}_0 \prec \mathfrak{m}_1 \prec \mathfrak{m}_2 \prec \cdots \qquad \text{in } \mathfrak{S}.$$

Denote a function $f : \mathfrak{M} \to \mathbb{R}$ as a series $\sum_{\mathfrak{m}} f_{\mathfrak{m}} \mathfrak{m}$ where $f_{\mathfrak{m}} = f(\mathfrak{m})$, with **support**

$$\operatorname{supp} f := \{\mathfrak{m} : f_{\mathfrak{m}} \neq 0\} \subseteq \mathfrak{M}.$$

Then

$$\mathbb{R}[[\mathfrak{M}]] \ := \ \{f : \mathfrak{M} \to \mathbb{R} : \ \operatorname{supp} f \text{ is well-based}\}$$

is a subspace of the $\mathbb{R}$-linear space $\mathbb{R}^{\mathfrak{M}}$. For $f \in \mathbb{R}[[\mathfrak{M}]]^{\neq}$ let

$$\mathfrak{d}(f) := \max \operatorname{supp} f$$

be the **dominant monomial** of f.

From now on assume $(\mathfrak{M}, \cdot, \preccurlyeq)$ is an ordered abelian group.

Then, with multiplication of well-based series defined by

$$f \cdot g \;=\; \sum_{\mathfrak{m}} \left(\sum_{\mathfrak{m}_1 \cdot \mathfrak{m}_2 = \mathfrak{m}} f_{\mathfrak{m}_1} \cdot g_{\mathfrak{m}_2} \right) \mathfrak{m},$$

we obtain an $\mathbb{R}$-algebra $\mathbb{R}[[\mathfrak{M}]]$.

Example

Take a multiplicative copy $x^{\mathbb{R}}$ of $(\mathbb{R}, \leqslant, 0, +)$ with isomorphism

$$r \mapsto x^r : \mathbb{R} \to x^{\mathbb{R}}.$$

Then $x^{\mathbb{R}}$ has the ordered subgroups $x^{\mathbb{Z}} \subseteq x^{\mathbb{Q}}$, and

$$\mathbb{R}(\!(x^{-1})\!) \;=\; \mathbb{R}[[x^{\mathbb{Z}}]] \;\subseteq\; \mathrm{P}(\mathbb{R}) \;\subseteq\; \mathbb{R}[[x^{\mathbb{Q}}]] \;\subseteq\; \mathbb{R}[[x^{\mathbb{R}}]].$$

A family (f_λ) in $\mathbb{R}[[\mathfrak{M}]]$ is said to be **summable** if

1. $\bigcup_\lambda \operatorname{supp} f_\lambda$ is well-based; and
2. for all $\mathfrak{m}$ there are only finitely many λ with $\mathfrak{m} \in \operatorname{supp} f_\lambda$.

We then define its sum $f = \sum_\lambda f_\lambda \in \mathbb{R}[[\mathfrak{M}]]$ by $f_\mathfrak{m} = \sum_\lambda f_{\lambda,\mathfrak{m}}$.

Examples

1. Given $f \in \mathbb{R}[[\mathfrak{M}]]$, the family $(f_\mathfrak{m}\mathfrak{m})$ is summable with sum f.
2. If $f \prec 1$, then (f^n) is summable with sum $\dfrac{1}{1-f}$.

Summability has various nice properties (e.g., rearrangement).

As a consequence of 2, $\mathbb{R}[[\mathfrak{M}]]$ is a field: write $f \in \mathbb{R}[[\mathfrak{M}]]^{\neq}$ as

$$f = c\,\mathfrak{m}\,(1-\varepsilon) \quad \text{where } c \in \mathbb{R}^\times,\ \mathfrak{m} \in \mathfrak{M},\ \varepsilon \prec 1;$$

then $f^{-1} = c^{-1}\mathfrak{m}^{-1}\sum_n \varepsilon^n.$

Turn $\mathbb{R}[[\mathfrak{M}]]$ into an ordered valued field satisfying, for $f \neq 0$:

$$f > 0 \quad \Longleftrightarrow \quad f_{\mathfrak{d}(f)} > 0,$$
$$f \prec 1 \quad \Longleftrightarrow \quad \mathfrak{d}(f) \prec 1.$$

The ordered valued field extension $\mathbb{R}[[\mathfrak{M}]]$ of $\mathbb{R}$ is called a **Hahn field**. Recall the valuation ring, its maximal ideal, and the residue field:

$$\mathcal{O} := \{f : f \preccurlyeq 1\}, \qquad \scriptstyle{\mathcal{O}} \displaystyle := \{f : f \prec 1\}, \qquad \boldsymbol{k} := \mathcal{O}/\scriptstyle{\mathcal{O}}.$$

The residue morphism $\mathcal{O} \to \boldsymbol{k}$ restricts to an isomorphism $\mathbb{R} \xrightarrow{\cong} \boldsymbol{k}$. The valuation $\mathbb{R}[[\mathfrak{M}]]^{\times} \xrightarrow{v} \Gamma$ restricts to an isomorphism $\mathfrak{M} \xrightarrow{\cong} \Gamma$ of groups, with

$$\mathfrak{m} \preccurlyeq \mathfrak{n} \quad \Longleftrightarrow \quad v\mathfrak{m} \geqslant v\mathfrak{n}.$$

Next we consider directed unions of Hahn fields: the ordered field $\mathbb{T}$ will be obtained as such a union.

Directed unions of Hahn fields

Let $(\mathfrak{M}_i)_{i\in I}$ with $I \neq \emptyset$ be a family of ordered subgroups of $\mathfrak{M}$ satisfying $\mathfrak{M} = \bigcup_i \mathfrak{M}_i$. Assume that $(\mathfrak{M}_i)$ is *directed*:

for all i, j there is k with $\mathfrak{M}_i, \mathfrak{M}_j \subseteq \mathfrak{M}_k$.

We then obtain the ordered valued subfield

$$K \ := \ \bigcup_i \mathbb{R}[[\mathfrak{M}_i]] \ \subseteq \ \mathbb{R}[[\mathfrak{M}]].$$

Example: $$\mathrm{P}(\mathbb{R}) \ = \ \bigcup_{n\geqslant 1} \mathbb{R}[[x^{(1/n)\mathbb{Z}}]] \ \subseteq \ \mathbb{R}[[x^{\mathbb{Q}}]].$$

A family (f_λ) in K is **summable** if there is an ordered subgroup $\mathfrak{G} \subseteq \mathfrak{M}$ such that $\mathbb{R}[[\mathfrak{G}]] \subseteq K$, all $f_\lambda \in \mathbb{R}[[\mathfrak{G}]]$, and (f_λ) is summable as a family in $\mathbb{R}[[\mathfrak{G}]]$; then $\sum_\lambda f_\lambda \in K$ is defined.
(NB: if I is countable, then each such $\mathfrak{G}$ is contained in some $\mathfrak{M}_i$.)

An $\mathbb{R}$-linear map $\Phi\colon K \to L$ is **strongly linear** if for every summable family (f_λ) in K the family $\big(\Phi(f_\lambda)\big)$ is summable in L, and

$$\Phi\big(\textstyle\sum_\lambda f_\lambda\big) = \sum_\lambda \Phi(f_\lambda).$$

E.g., given $g \in K$, the operator $f \mapsto fg$ on K is strongly linear.

Let $t = (t_1, \ldots, t_n)$ be a tuple of distinct variables and let

$$F = F(t) = \sum_{\nu} F_\nu t^\nu \in \mathbb{R}[[t_1, \ldots, t_n]]$$

be a formal power series over $\mathbb{R}$. Here

$$\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n, \qquad F_\nu \in \mathbb{R}, \qquad t^\nu := t_1^{\nu_1} \cdots t_n^{\nu_n}.$$

For any tuple $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ of elements of $\mathcal{O}_K$ the family $(F_\nu \varepsilon^\nu)$ is summable, where $\varepsilon^\nu := \varepsilon_1^{\nu_1} \cdots \varepsilon_n^{\nu_n}$ ("Neumann's Lemma"). Put

$$F(\varepsilon) := \sum_{\nu} F_\nu \varepsilon^\nu \in \mathcal{O}_K.$$

Using Taylor expansions this allows us to extend each restricted analytic function $\mathbb{R}^n \to \mathbb{R}$ to a map $K^n \to K$, and hence turn K into an extension of the $\mathcal{L}_{\text{an}}$-structure $\mathbb{R}_{\text{an}}$. If all $\mathfrak{M}_i$ are *divisible*, this is an *elementary* extension.

However, when trying to define an extension of the real exponential function to Hahn fields we run into problems:

An **exponential ordered field** is an ordered field E equipped with an exponentiation, that is, an embedding

$$\exp\colon (E, +, \leqslant) \to (E^{>}, \cdot, \leqslant).$$

If $\exp(E) = E^{>}$ then we call E a **logarithmic-exponential ordered field**, and denote the inverse of $\exp$ by $\log\colon E^{>} \to E$.

Examples

The ordered field $\mathrm{Li}(\mathbb{R})$ with exponentiation $f \mapsto \mathrm{e}^{f}$, and its logarithmic-exponential ordered subfields H_{LE} and $\mathbb{R}$.

We can't turn $\mathbb{R}[[x^{\mathbb{R}}]]$ into a log-exp ordered field.
(Kuhlmann-Kuhlmann-Shelah: not even $\mathbb{R}[[\mathfrak{M}]]$ when $\mathfrak{M} \neq \{1\}$.)

To remedy this, we extend $\mathbb{R}[[x^{\mathbb{R}}]]$ in two steps:

1. first close off under $\exp$ to obtain the exponential ordered field $\mathbb{T}_{\exp}$ of **exponential transseries**;
2. then close off under $\log$ to arrive at the log-exp ordered field $\mathbb{T}$ of **transseries**.

Let $(E, A, B, \exp)$ be a **pre-exponential ordered field**:

1. E is an ordered field;
2. $A, B \subseteq E$ are additive subgroups of E such that $E = A \oplus B$ and B convex in E;
3. $\exp$ is an ordered group embedding $(B, +, \leqslant) \to (E^{>}, \cdot, \leqslant)$.

(Think of $\exp$ as a partially defined exponentiation.)

Example

$E = \mathbb{R}[[x^{\mathbb{R}}]], \quad A = \mathbb{R}[[x^{\mathbb{R}^{>}}]], \quad B = \mathcal{O}_E = \mathbb{R} \oplus o_E,$

$$\exp(r + \varepsilon) = \mathrm{e}^r \cdot \sum_n \frac{\varepsilon^n}{n!} \qquad (r \in \mathbb{R},\ \varepsilon \prec 1).$$

In the following we suppose $E = \mathbb{R}[[\mathfrak{M}]]$.

Construction of $\mathbb{T}_{\exp}$

We then define a pre-exponential ordered field $(E^*, A^*, B^*, \exp^*)$ extending $(E, A, B, \exp)$ such that $E \subseteq B^* =$ domain of $\exp^*$:

1. Take an ordered group isomorphism $\exp^*: A \to \exp^*(A)$ onto a *multiplicative* copy of A. Order $\mathfrak{M}^* := \mathfrak{M} \times \exp^*(A)$ so that $\mathfrak{M}$ and $\exp^*(A)$ are ordered subgroups of $\mathfrak{M}^*$ and $\mathfrak{M}$ is convex in $\mathfrak{M}^*$.

---

2. Set $E^* := \mathbb{R}[[\mathfrak{M}^*]] = \mathbb{R}[[\mathfrak{M} \exp^*(A)]] \supseteq E = \mathbb{R}[[\mathfrak{M}]]$.

---

3. Put $\mathcal{O}^* := \mathbb{R}[[\mathfrak{M} \exp^*(A^<)]]$. With

$$A^* := \mathbb{R}[[\mathfrak{M} \exp^*(A^>)]],$$

$$B^* := \mathbb{R}[[\mathfrak{M} \exp^*(A^\leqslant)]] = E \oplus \mathcal{O}^* = A \oplus B \oplus \mathcal{O}^*$$

we have $E^* = A^* \oplus B^*$ and B^* is convex in E^*.

---

4. Extend $\exp^*$ to $\exp^*: B^* \to (E^*)^>$ by

$$\exp^*(a+b+\varepsilon) := \exp^*(a) \cdot \exp(b) \cdot \sum_n \frac{\varepsilon^n}{n!} \quad (a \in A,\ b \in B,\ \varepsilon \in \mathcal{O}^*).$$

Recursively define

$$(E_0, A_0, B_0, \exp_0) \;:=\; (\mathbb{R}[[x^{\mathbb{R}}]], \ldots),$$
$$(E_{n+1}, A_{n+1}, B_{n+1}, \exp_{n+1}) \;:=\; (E_n^*, A_n^*, B_n^*, \exp_n^*),$$

and put

$$\mathbb{T}_{\exp} \;=\; \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{E}} \;:=\; \bigcup_n E_n.$$

Then $\mathbb{T}_{\exp}$ equipped with the common extension $\exp\colon \mathbb{T}_{\exp} \to \mathbb{T}_{\exp}^{>}$ of the $\exp_n$ is an exponential ordered field extension of $\mathbb{R}$.

☹ $\mathbb{T}_{\exp}$ is not a *logarithmic*-exponential ordered field:

$$\mathbb{T}_{\exp}^{>} = x^{\mathbb{R}} \cdot \exp(\mathbb{T}_{\exp}) \quad \text{with} \quad x \notin \exp(\mathbb{T}_{\exp}).$$

Idea

Successively replace x by new variables $\ell_1 = \log x$, $\ell_2 = \log\log x$, …

Formally, we introduce a strongly linear isomorphism

$$f \mapsto f{\downarrow_n} = f(\ell_n) \colon \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{E}} \xrightarrow{\cong} \mathbb{R}[[\ell_n^{\mathbb{R}}]]^{\mathrm{E}}$$

of ordered exponential fields.

We identify $\mathbb{R}[[\ell_n^{\mathbb{R}}]]^{\mathrm{E}}$ with its image under the strongly linear exponential ordered field embedding

$$\mathbb{R}[[\ell_n^{\mathbb{R}}]]^{\mathrm{E}} \hookrightarrow \mathbb{R}[[\ell_{n+1}^{\mathbb{R}}]]^{\mathrm{E}} \quad \text{with} \quad \ell_n^r \mapsto \exp(r\ell_{n+1}) \text{ for each } r \in \mathbb{R}.$$

So we have inclusions

$$\mathbb{T}_{\exp} \;=\; \mathbb{R}[[\ell_0^{\mathbb{R}}]]^{\mathrm{E}} \;\subseteq\; \mathbb{R}[[\ell_1^{\mathbb{R}}]]^{\mathrm{E}} \;\subseteq\; \mathbb{R}[[\ell_2^{\mathbb{R}}]]^{\mathrm{E}} \;\subseteq\; \cdots$$

of exponential ordered fields, and we obtain the log-exp ordered field

$$\mathbb{T} \;=\; \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}} \;:=\; \textstyle\bigcup_n \mathbb{R}[[\ell_n^{\mathbb{R}}]]^{\mathrm{E}}.$$

By construction, $\mathbb{T}$ is an increasing union of increasing unions of Hahn fields; it can also be represented as a directed union of Hahn fields.

Upward and downward shift

These are the unique strongly linear automorphism $f \mapsto f{\uparrow}$ of the exponential ordered field $\mathbb{T}$ that sends x to e^x, with inverse $f \mapsto f{\downarrow}$. Their nth iterates are $f \mapsto f{\uparrow^n}$, respectively, $f \mapsto f{\downarrow_n}$.

Thus $x{\downarrow_n} = \ell_n$, and for each $f \in \mathbb{T}$ there is an n with $f{\uparrow_n} \in \mathbb{T}_{\exp}$.

Example (relevant for later)

$$\lambda := \frac{1}{\ell_0} + \frac{1}{\ell_0 \ell_1} + \frac{1}{\ell_0 \ell_1 \ell_2} + \cdots \notin \mathbb{T}.$$

The sequence

$$\cdots < \ell_2 < \ell_1 < \ell_0 = x = \mathrm{e}_0 < \mathrm{e}_1 < \mathrm{e}_2 < \cdots$$

is coinitial and cofinal in $\mathbb{T}^{>\mathbb{R}} = \{f \in \mathbb{T} : f > \mathbb{R}\}$.

Fact

If $\mathfrak{M}$ is divisible, then the ordered Hahn field $\mathbb{R}[[\mathfrak{M}]]$ is real closed.

As a consequence, $\mathbb{T}$ is real closed, and thus (by Tarski), an elementary extension of the ordered field $\mathbb{R}$.

Indeed, by work of van den Dries-Macintyre-Marker (1994), $\mathbb{T}$ is also an elementary extension of the $\mathcal{L}_{\mathrm{an,exp}}$-structure $\mathbb{R}$; in particular, of the exponential ordered field $\mathbb{R}$.

But $\mathbb{T}$ also has a *differential* structure:

A derivation on a field K is a map $\partial\colon K \to K$ such that

$$\partial(f+g) = \partial(f) + \partial(g), \quad \partial(fg) = \partial(f)g + f\partial(g) \quad \text{for all } f, g \in K.$$

A differential field is a field K with a derivation ∂ on K. The constant field of a differential field K is the subfield $\ker \partial$ of K.

Theorem (Écalle, v. d. Dries-Macintyre-Marker, v. d. Hoeven, 1990s)

There is a unique strongly linear derivation $f \mapsto f'$ on $\mathbb{T}$ such that

$$x' = 1 \qquad \textit{and} \qquad (\exp f)' = f' \exp f \textit{ for all } f \in \mathbb{T}.$$

The differential field $\mathbb{T}$ has some nice (but non-trivial) properties:

1. its constant field is $\mathbb{R}$;
2. every $f \in \mathbb{T}$ has an antiderivative in $\mathbb{T}$.

($\Longrightarrow$ for all $g, h \in \mathbb{T}$ there is a $y \in \mathbb{T}^{\times}$ with $y' + gy = h$.)

1. The inverse of $\mathrm{e}^x + x$

$$\frac{1}{\mathrm{e}^x + x} = \frac{1}{\mathrm{e}^x(1 + x\,\mathrm{e}^{-x})} = \mathrm{e}^{-x} \sum_n (-1)^n (x\,\mathrm{e}^{-x})^n$$

2. The logarithm of $\sinh = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})$

$$\log(\sinh) = \log\left(\frac{\mathrm{e}^x}{2}(1 - \mathrm{e}^{-2x})\right) = x - \log 2 - \sum_{n \geqslant 1} \frac{\mathrm{e}^{-2nx}}{n}$$

3. Integrating $1/\log x$

$$\left(\frac{x}{\log x} \sum_n \frac{n!}{(\log x)^n}\right)' = \frac{1}{\log x}$$

Theorem (A.-v. d. Dries, 2002)

Let H be a Hardy field with $H \supseteq \mathbb{R}$. Then every embedding $H \to \mathbb{T}$ of ordered differential fields extends to an embedding $\operatorname{Li}(H) \to \mathbb{T}$.

Thus there is an embedding of $\operatorname{Li}(\mathbb{R})$, and hence of H_{LE}, into $\mathbb{T}$.

We may view an embedding $H \to \mathbb{T}$ as a *formal expansion operator* and its inverse as a *summation operator.*

Example

The germ of the error function

$$\operatorname{erf}\colon \mathbb{R} \to \mathbb{R}, \quad \operatorname{erf}(t) := \frac{2}{\sqrt{\pi}} \int_0^t \mathrm{e}^{-s^2}\, ds$$

lies in $\operatorname{Li}(\mathbb{R})$. Any embedding $\operatorname{Li}(\mathbb{R}) \to \mathbb{T}$ maps it onto the transseries

$$1 - \frac{\mathrm{e}^{-x^2}}{x\sqrt{\pi}} \left(1 + \sum_{n \geqslant 1} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2x^2)^n} \right).$$

Embedding Hardy fields into $\mathbb{T}$

E. Kaplan (2022) has generalized the theorem above. A special case: Say that a Hardy field H is $\mathcal{L}_{\text{an}}$**-closed** if for each restricted analytic function $F\colon \mathbb{R}^n \to \mathbb{R}$ and $g_1, \dots, g_n \in H$, the germ of the function

$$t \mapsto F\big(g_1(t), \dots, g_n(t)\big)$$

is also in H. We then turn H into an $\mathcal{L}_{\text{an}}$-structure in the natural way, and if H is also closed under $\exp$, to an $\mathcal{L}_{\text{an},\exp}$-structure.
Let

$$\mathrm{Li}(\mathbb{R})_{\text{an}} := \begin{cases} \text{the smallest } \mathcal{L}_{\text{an}}\text{-closed Hardy field} \\ \text{which is closed under } \exp \text{ and } \textstyle\int. \end{cases}$$

There is an embedding $\mathrm{Li}(\mathbb{R})_{\text{an}} \to \mathbb{T}$ of ordered differential fields which is also a morphism of $\mathcal{L}_{\text{an},\exp}$-structures.

($\Longrightarrow$ the Hardy field of germs of functions definable in $\mathbb{R}_{\text{an},\exp}$ embeds into $\mathbb{T}$: van den Dries-Macintyre-Marker.)

Some questions about $\mathbb{T}$

The differential field $\mathbb{T}$ turned out to be amenable to a computational treatment: J. van der Hoeven (~ 2000) gave a quasi-algorithmic method for solving algebraic differential equations like

$$P(y, y', \ldots, y^{(n)}) = 0 \qquad (P \in \mathbb{R}[Y_0, Y_1, \ldots, Y_n])$$

in $\mathbb{T}$. This motivates the questions:

1. Can we do something similar in Hardy fields?
2. Even more ambitiously: what are the first-order logical properties of the differential field $\mathbb{T}$? Or of "sufficiently rich" Hardy fields?
3. Can we construct expansion/summation operators encompassing (exponentially bounded) Hardy fields bigger than $\mathrm{Li}(\mathbb{R})$?

In later lectures we will see some answers to these questions. Before we finish, we point out some further structure on $\mathbb{T}$.

Let f range over $\mathbb{T}$ and g over $\mathbb{T}^{>\mathbb{R}}$.

Theorem (see v. d. Dries-Macintyre-Marker, 2001)

There is a unique operation

$$(f, g) \mapsto f \circ g \ : \ \mathbb{T} \times \mathbb{T}^{>\mathbb{R}} \to \mathbb{T}$$

such that for each g, the map $f \mapsto f \circ g \colon \mathbb{T} \to \mathbb{T}$ is a strongly linear embedding of exponential ordered fields with $x \circ g = g$.

This operation satisfies $f \circ x = f$ and $f \circ \mathrm{e}^x = f\!\uparrow$, and obeys

$$(f \circ g)' = (f' \circ g) \cdot g' \qquad \text{(Chain Rule).}$$

Moreover, $\circ$ turns $\mathbb{T}^{>\mathbb{R}}$ into a group with identity element x.

V. Bagayoko has started to explore the solution sets of one-variable equations in this and related groups.

Example (*Lambert W function*: compositional inverse of $x\,\mathrm{e}^x$)

An asymptotic expansion for W is given by a transseries

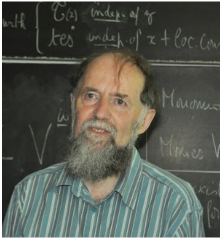
$$\begin{aligned} W &\sim \log x - \log_2 x + \sum_{m\geqslant 0,\ n\geqslant 1} c_{mn} \frac{(\log_2 x)^n}{(\log x)^{m+n}} \\ &= \log x - \log_2 x + \frac{\log_2 x}{\log x} + \\ &\qquad \frac{(\log_2 x)^2}{2(\log x)^2} - \frac{\log_2 x}{(\log x)^2} + \\ &\qquad \frac{(\log_2 x)^3}{3(\log x)^3} - \frac{3(\log_2 x)^2}{2(\log x)^3} + \frac{\log_2 x}{(\log x)^3} + \cdots \end{aligned}$$

for certain coefficients $c_{mn} \in \mathbb{Q}$ which are given by an explicit formula, and $\log_2 x = \log\log x\ [= \ell_2]$. (de Bruijn, Comtet)

(This transseries is actually absolutely convergent for large $x \to \infty$.)

L. Euler
(1707–1783)

H. Hahn
(1879–1934)



J. Écalle
(1950–)

Tutorial on Hardy fields and transseries

Part III: Asymptotic differential algebra

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Plan for my lectures

I Hardy fields
II Transseries
III Asymptotic differential algebra
IV Model theory of transseries
V Maximal Hardy fields

In the last lectures, we met a variety of interesting differential fields (of transseries, germs of functions, ...) equipped with asymptotic structure, such as *ordering* and *dominance*.

We will now introduce an algebraic framework which unifies these examples and helps to unravel their model-theoretic properties.

All based on joint work with (one or both of)

L. van den Dries and J. van der Hoeven.

III. Asymptotic differential algebra

Let K be a differential field (of characteristic 0), with derivation ∂.

As usual

$$f' = \partial(f),\ f'' = \partial^2(f),\ \ldots,\ f^{(n)} = \partial^n(f),\ \ldots$$

The **constant field** of K is $C = C_K := \ker\partial = \{f \in K : f' = 0\}$.

For $f \neq 0$ let $f^\dagger := f'/f$ be the **logarithmic derivative** of f. Note

$$(f \cdot g)^\dagger = f^\dagger + g^\dagger \qquad \text{for } f, g \neq 0.$$

The ring of **differential polynomials** (= d-polynomials) in $Y_1, \ldots, Y_n$ with coefficients in K is denoted by $K\{Y_1, \ldots, Y_n\}$. E.g.:

$$P(Y) = Y(Y')^2 + Y''Y^{(5)} - 1 \in \mathbb{Q}\{Y\} \qquad (n = 1,\ Y = Y_1).$$

The setting of *ordered* differential algebra

An **ordered differential field** is a differential field K equipped with an ordering making it an ordered field. We can then also turn K into a *valued field* with dominance relation

$$f \preccurlyeq g \quad :\Longleftrightarrow \quad |f| \leqslant c|g| \text{ for some } c \in C.$$

Examples

1. Each Hardy field H is an ordered differential field with $C_H \subseteq \mathbb{R}$, and for $g \neq 0$, we have:

$$f \preccurlyeq g \iff \lim_{t \to +\infty} \frac{f(t)}{g(t)} \in \mathbb{R}, \quad f \prec g \iff \lim_{t \to +\infty} \frac{f(t)}{g(t)} = 0,$$
$$f \asymp g \iff \lim_{t \to +\infty} \frac{f(t)}{g(t)} \in \mathbb{R}^\times, \quad f \sim g \iff \lim_{t \to +\infty} \frac{f(t)}{g(t)} = 1.$$

2. The dominance relation $\preccurlyeq$ on the ordered differential field $\mathbb{T}$ from above agrees with the one on $\mathbb{T}$ *qua* valued subfield of a Hahn field $\mathbb{R}[[\mathfrak{M}]]$.

The $\mathbb{T}$-Conjecture

This is the idea that $\mathbb{T}$ is a "universal" domain for asymptotic differential algebra:

> [*The differential field* $\mathbb{T}$] *marks an almost impassable horizon for "ordered analysis".* (*This sector of analysis is in some sense "orthogonal" to harmonic analysis.*) —J. Écalle

To formulate a precise version, we view ordered valued differential fields model-theoretically as structures with the primitives

$$0,\ 1,\ +,\ \times,\ \partial \text{ (derivation)},\quad \leqslant \text{ (ordering)},\quad \preccurlyeq \text{ (dominance)}.$$

The $\mathbb{T}$-Conjecture

$\mathbb{T}$ is model complete.

(The inclusion of $\preccurlyeq$ is necessary.)

Model completeness of $\mathbb{T}$ can be expressed geometrically in terms of systems of algebraic differential (in)equations. (Similar to Gabrielov's "theorem of the complement" for real subanalytic sets.)

For this, define a d**-algebraic** set in $\mathbb{T}^n$ to be a zero set

$$\{y \in \mathbb{T}^n : P_1(y) = \cdots = P_m(y) = 0\}$$

of some d-polynomials $P_1, \ldots, P_m \in \mathbb{T}\{Y_1, \ldots, Y_n\}$.

An H**-algebraic** set in $\mathbb{T}^n$ is the intersection of a d-algebraic set in $\mathbb{T}^n$ with a set

$$\{(y_1, \ldots, y_n) \in \mathbb{T}^n : y_i \prec 1 \text{ for all } i \in I\} \quad \text{where } I \subseteq \{1, \ldots, n\}.$$

The image of an H-algebraic set in $\mathbb{T}^n$, for some $n \geqslant m$, under the natural projection $\mathbb{T}^n \to \mathbb{T}^m$ is called **sub-H-algebraic.**

Model completeness of $\mathbb{T}$ means (almost):

> *the complement of any sub-H-algebraic set in $\mathbb{T}^m$ is again sub-H-algebraic.*

(A strengthening of model completeness is *quantifier elimination*: it describes sub-H-algebraic sets using additional primitives on $\mathbb{T}$.)

To prove model completeness results algebraically, we need to develop an extension theory for structures with the same basic universal properties as the structure of interest.

This strategy can be employed to analyze the logical properties of classical fields like $\mathbb{C}$, $\mathbb{R}$, $\mathbb{C}((t))$, $\ldots$

We do something similar for $\mathbb{T}$.

For this we define the class of H-fields (H: Hardy, Hausdorff, Hahn.)

The goal then is to show that the class of *existentially closed* H-fields is axiomatizable in first-order logic, and contains $\mathbb{T}$.

If successful, we have model completeness of $\mathbb{T}$.

Here, an H-field H is **existentially closed** if every system of algebraic differential equations and asymptotic conditions which has a solution in some H-field extension of H also has a solution in H.

H-fields

These are ordered differential fields in which ordering, dominance, and derivation interact in a certain nice way:

Definition

Let H be an ordered differential field with constant field $C = C_H$. Then H is an H**-field** if

(H1) $f \succ 1 \;\Rightarrow\; f^\dagger > 0$;

(H2) $f \asymp 1 \;\Rightarrow\; f \sim c$ for some $c \in C^\times$;

(H3) $f \prec 1 \;\Rightarrow\; f' \prec 1$.

(The usual definition of "H-field" doesn't include (H3).)

Examples

- every Hardy field $H \supseteq \mathbb{R}$;
- the ordered differential field $\mathbb{T}$;
- each ordered differential subfield of an H-field H containing C.

H-fields are part of the (more flexible) category of "differential-valued fields" of Rosenlicht (1980s).

Many basic properties of the dominance relation valid in Hardy fields are consequences of the H-field axioms.

For example, let H be an H-field; then:

l'Hôpital's Rule

If $0 \neq f, g \not\asymp 1$, then $f \preccurlyeq g \iff f' \preccurlyeq g'$.

Besides being real closed H-fields, $\mathbb{T}$ and $\mathrm{Li}(\mathbb{R})$ are Liouville closed:

We call a real closed H-field **Liouville closed** if it satisfies

$$\forall f, g\, \exists y \big[y \neq 0 \;\&\; y' + fy = g \big].$$

A **Liouville closure** of an H-field H is a minimal Liouville closed H-field extension of H.

Theorem (A.-v. d. Dries, 2002)

Every H-field H has exactly one or exactly two Liouville closures, up to isomorphism over H.

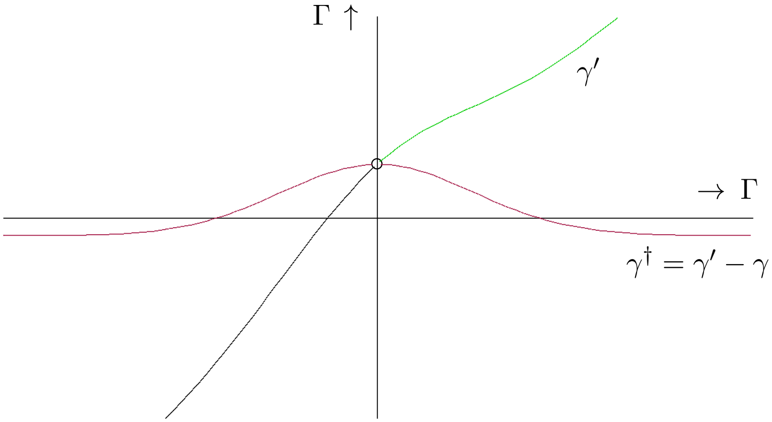
What can go wrong when forming Liouville closures may be seen from the *asymptotic couple* of H.

To explain this, let $v\colon H^\times \to \Gamma$ be the valuation of H. We have a map

$$\gamma = vg \mapsto \gamma' = v(g') \colon \quad \Gamma^{\neq} = \Gamma \setminus \{0\} \to \Gamma.$$

(As a consequence of l'Hôpital.)

The pair consisting of Γ and the map $\gamma \mapsto \gamma^\dagger := \gamma' - \gamma$ is called the **asymptotic couple** of H. Always $(\Gamma^{\neq})^\dagger < (\Gamma^{>})'$.



Exactly one of the following statements holds:

1. $(\Gamma^{\neq})^{\dagger} < \gamma < (\Gamma^{>})'$ *for a (necessarily unique)* γ.
   We call such γ a **gap** in H.
2. $(\Gamma^{\neq})^{\dagger}$ *has a largest element.*
   We say that H is **grounded**.
3. $(\Gamma^{\neq})^{\dagger}$ *has no supremum; equivalently:* $\Gamma = (\Gamma^{\neq})'$.
   We say that H has **asymptotic integration.**

Examples

1. $H = C$;
2. $H = \mathbb{T}_{\exp}$;
3. $H = \mathbb{T}$ (or any other Liouville closed H).

Exactly one of the following statements holds:

1. $(\Gamma^{\neq})^{\dagger} < \gamma < (\Gamma^{>})'$ *for a (necessarily unique)* γ.
   We call such γ a **gap** in H.
2. $(\Gamma^{\neq})^{\dagger}$ *has a largest element.*
   We say that H is **grounded**.
3. $(\Gamma^{\neq})^{\dagger}$ *has no supremum; equivalently:* $\Gamma = (\Gamma^{\neq})'$.
   We say that H has **asymptotic integration.**

In 1 we have *two* Liouville closures: if $\gamma = vg$, then we have a choice when adjoining $\int g$: make it $\succ 1$ or $\prec 1$.

In 2 we have *one* Liouville closure: if $vg = \max{(\Gamma^{\neq})^{\dagger}}$, then $\int g \succ 1$ in each Liouville closure of H.

In 3 we may have *one or two* Liouville closures.

Every H-subfield $H \supseteq \mathbb{R}$ of $\mathbb{T}$ has a unique Liouville closure.

The intrinsic reason for this: Suppose $H \supseteq \mathbb{R}(\ell_0, \ell_1, \dots)$ and $\gamma \in H$ (or rather $v\gamma$) is a gap in H: $(\Gamma^<)^\dagger < v\gamma < (\Gamma^>)'$

Then $v\ell_n$ is cofinal in $\Gamma^<$ and $v(1/\ell_n)$ is coinitial in $\Gamma^>$,
hence $v(\ell_n^\dagger)$ is cofinal in $(\Gamma^<)^\dagger$ and $v((1/\ell_n)')$ is coinitial in $(\Gamma^>)'$.

Now $\gamma_n := \ell_n^\dagger = \dfrac{1}{\ell_0 \cdots \ell_n}, \qquad (1/\ell_n)' = \gamma_n/\ell_n,$

hence $\gamma_n \succ \gamma \succ (1/\ell_n)' = \gamma_n/\ell_n$

and since $\lambda_n := -\gamma_n^\dagger = \dfrac{1}{\ell_0} + \dfrac{1}{\ell_0\ell_1} + \cdots + \dfrac{1}{\ell_0\ell_1\cdots\ell_n},$

we get

$$\lambda := -\gamma^\dagger = \frac{1}{\ell_0} + \frac{1}{\ell_0\ell_1} + \frac{1}{\ell_0\ell_1\ell_2} + \cdots + \frac{1}{\ell_0\ell_1\cdots\ell_n} + \cdots + \text{smaller terms} \quad \lightning$$

This fact about $\mathbb{T}$ translates into a $\forall\exists$-statement about H-fields:

Definition

An ungrounded H-field H is **λ-free** if there is no $\lambda \in H$ such that

$$\lambda - g^{\dagger\dagger} \prec g^{\dagger} \qquad \text{for each } g \succ 1 \text{ in } H.$$

Theorem (A. Gehret, 2017)

An H-field has a unique Liouville closure $\Longleftrightarrow$ it is grounded or λ-free.

This is a stronger, and more robust, property than λ-freeness.

Just like λ-freeness has to do with solving first-order linear differential equations, ω-freeness is connected to order 2 equations:

Examples (2nd order linear)

- $y'' = -y$ has *no* solution $y \in \mathbb{T}^\times$;
- $y'' = xy$ has *two* $\mathbb{R}$-linearly independent solutions in $\mathbb{T}$:

$$\mathrm{Ai} = \frac{\mathrm{e}^{-\xi}}{2\pi^{1/2}x^{1/4}} \sum_n (-1)^n \frac{a_n}{\xi^n}$$

$$\mathrm{Bi} = \frac{\mathrm{e}^{\xi}}{\pi^{1/2}x^{1/4}} \sum_n (-1)^n \frac{a_n}{\xi^n} \qquad (\xi = \tfrac{2}{3}x^{3/2},\ a_n \in \mathbb{R}).$$

Let H be a Liouville closed H-field. For $f \in H$ and $y \in H^\times$,

$$4y'' + fy = 0 \quad \Longleftrightarrow \quad -4y''/y = f \quad \Longleftrightarrow \quad \omega(2y^\dagger) = f$$

where $\boxed{\omega(z) := -(2z' + z^2)}$

(a relative of the *Schwarzian derivative*).

Hence

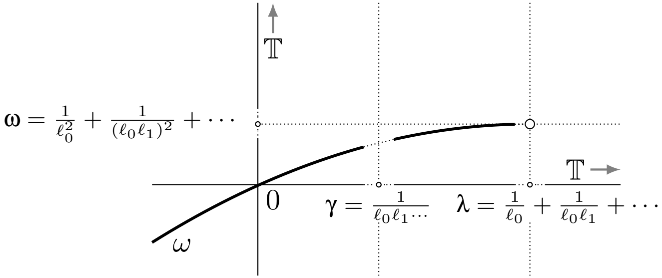
$$\omega(H) = \{ f \in H : 4y'' + fy = 0 \text{ for some } y \in H^\times \}.$$

The case $H = \mathbb{T}$

The sequence

$$\omega(\lambda_n) \;=\; \frac{1}{\ell_0^2} + \frac{1}{(\ell_0\ell_1)^2} + \cdots + \frac{1}{(\ell_0\ell_1\cdots\ell_n)^2}$$

is cofinal in $\omega(\mathbb{T})$. Thus $\omega(\mathbb{T})$ has no supremum in $\mathbb{T}$.



Once again, this can be translated into a statement about H-fields:

Definition

Call an ungrounded H-field H **ω-free** if there is no $\omega \in H$ with

$$\omega - \omega(-g^{\dagger\dagger}) \prec (g^\dagger)^2 \qquad \text{for each } g \succ 1 \text{ in } H.$$

ω-freeness is amazingly robust, and prevents deviant behavior:

If H is ω-free, then

- so is every differentially algebraic H-field extension of H;
- H is λ-free, and hence has only one Liouville closure; …

The main engine behind this:

Newton polynomials of one-variable differential polynomials over ω-free H have a very simple shape.

The definition of Newton polynomials relies on compositional conjugation.

Compositional conjugation

In $H = \mathbb{T}$ every differential polynomial $P \in H\{Y\}$ can be transformed, by applying finitely many transformations

$$f \;\mapsto\; f{\uparrow} = f \circ \mathrm{e}^x = f(\mathrm{e}^x) \qquad \text{(upward shift)},$$

into one with a "dominant part" of the form

$$(c_0 + c_1 Y + \cdots + c_m Y^m) \cdot (Y')^n \qquad (c_0, \ldots, c_m \in \mathbb{R}).$$

General H-fields H have no operation like $f \mapsto f{\uparrow}$.

But there is a substitute:

Compositional conjugation (in a d-field K with derivation ∂)

- Replacing ∂ by $\phi^{-1}\partial$ ($\phi \in K^\times$) yields a new d-field K^ϕ, and
- rewriting P in terms of $\phi^{-1}\partial$ yields $P^\phi \in K^\phi\{Y\}$ such that

$$P^\phi(y) = P(y) \quad \text{for all } y \in K.$$

Compositional conjugation

Compare with composition in a Hardy field H:

Reminder from Part I

Let $\ell \in H^{>\mathbb{R}}$ and $\phi := \ell'$. Then $\phi > 0$, and we have an ordered field isomorphism

$$h \mapsto h^\circ := h \circ \ell^{\mathrm{inv}} \colon H \to H^\circ.$$

This is not a *differential* field isomorphism: since

$$(h^\circ)' = (\phi^{-1} h')^\circ,$$

it is rather a differential field isomorphism $H^\phi \to H^\circ$.

The operation $P \mapsto P^\phi$ can be viewed as a **triangular** automorphism of the K-algebra $K\{Y\} = K[Y, Y', \ldots] = K^\phi\{Y\}$:

$$\begin{aligned} Y^\phi &= Y \\ (Y')^\phi &= \phi Y' \\ (Y'')^\phi &= \phi^2 Y'' + \phi' Y' \\ (Y''')^\phi &= \phi^3 Y''' + 3\phi\phi' Y'' + \phi'' Y', \\ &\vdots \end{aligned}$$

Such triangular automorphisms can be treated with Lie theoretic methods: every triangular automorphism σ of $K\{Y\}$ can be represented by an upper triangular matrix $M_\sigma \in K^{\mathbb{N}\times\mathbb{N}}$, whose matrix logarithm $\log(M_\sigma)$ represents a K-linear derivation of $K\{Y\}$.

Newton polynomials

Suppose now H is an ungrounded H-field.

We then only use **active** ϕ, those for which H^ϕ is again an H-field:

$$\phi > 0 \text{ and } \phi \succ h' \text{ for all } h \prec 1.$$

Theorem

Let $P \in H\{Y\}^{\neq}$. Then there exists $N_P \in C\{Y\}^{\neq}$ so that for all sufficiently small ϕ:

$$P^\phi = \mathfrak{d}\, N_P + R, \qquad \mathfrak{d} \in H^\times,\ R \prec \mathfrak{d}.$$

We call N_P the **Newton polynomial** *of P.*

(We omitted technical hypotheses to make N_P well-defined.)

It is not always the case (like in $\mathbb{T}$) that $N_P \in C[Y](Y')^{\mathbb{N}}$. Consider

$$P = N - \omega \cdot (Y')^2 \quad \text{where } N := 2Y'Y''' - 3(Y'')^2.$$

Then for $\phi = \gamma_n = \frac{1}{\ell_0 \ell_1 \ldots \ell_n}$:

$$P^\phi = \phi^4 N + \underbrace{(2\phi\phi'' - 3(\phi')^2 - \omega\phi^2)}_{(\omega(\lambda_n) - \omega)\phi^2} \cdot (Y')^2$$

$$\text{where} \quad \omega(\lambda_n) - \omega = \frac{1}{(\ell_0 \cdots \ell_{n+1})^2} + \cdots \prec \phi^2$$

and so $P^\phi \sim \phi^4 N$, thus $N_P = N$. However (!):

Theorem

H ω-free $\iff$ $N_P \in C[Y](Y')^{\mathbb{N}}$ *for all* $P \in H\{Y\}^{\neq}$.

The proof relies on the Lie algebra approach to compositional conjugation mentioned above.

The **Newton degree** of P is defined as $\boxed{\operatorname{ndeg} P \ := \ \deg N_P.}$

If H is ω-free, then N_P (and hence $\operatorname{ndeg} P$) doesn't change if we pass from H to an H-field extension.

Definition

We say that H is **newtonian** if every $P \in H\{Y\}^{\neq}$ with $\operatorname{ndeg} P = 1$ has a zero $y \preccurlyeq 1$ in H.

$\mathbb{T}$ is newtonian (as a directed of grounded Hahn H-subfields).

This is *the* most significant asymptotic-differential-algebraic property of $\mathbb{T}$, and the appropriate differential version of the henselian property of valuation theory.

It guarantees, for example, that the Painlevé II equation

$$y'' = 2y^3 + xy + \alpha \qquad (\alpha \in \mathbb{R})$$

has a solution in $y \preccurlyeq 1$ in $\mathbb{T}$: with

$$P := Y'' - 2Y^3 - xY - \alpha \in \mathbb{T}\{Y\},$$

for $\phi \prec 1$ we have

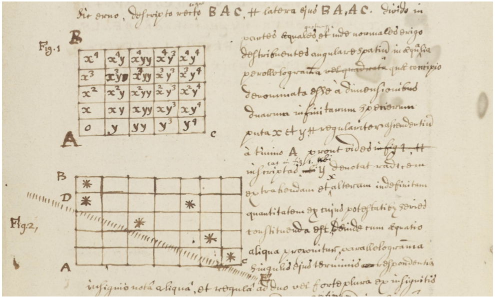
$$P^{\phi} = \phi^2 Y'' + \phi' Y' - 2Y^3 - xY - \alpha \quad \text{where } \phi^2, \phi' \prec 1 \prec x.$$

Thus $N_P \in \mathbb{R}^{\times} Y$, so $\operatorname{ndeg} P = 1$.

(It is known that P has a zero $y \preccurlyeq 1$ in $\mathbb{R}(x) \subseteq \mathbb{T}$ iff $\alpha \in \mathbb{Z}$.)

Newtonianity

We chose the adjective "newtonian" since this property allows us to develop a Newton diagram method for differential polynomials over H-fields (among others).

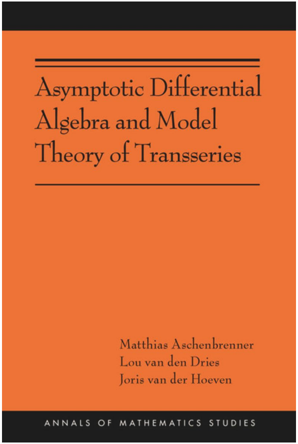


(from a letter of Isaac Newton to Henry Oldenburg, Oct. 24, 1676)

Theorem (sample application of differential Newton diagrams)

Every odd-degree differential polynomial over a real closed ω-free newtonian H-field has a zero.

For all this (and more), see our book $\longrightarrow$



Don't forget to check out
`https://tinyurl.com/ADH-errata`

Outlook

In the next lecture we will discuss further properties and uses of newtonianity.

We will then combine the three crucial features of $\mathbb{T}$,

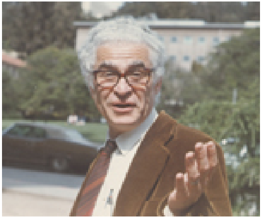
Liouville closedness, ω-*freeness*, and *newtonianity*

to unravel (a more precise version of) the $\mathbb{T}$-Conjecture, and describe its various consequences.

Then we will return to the world of Hardy fields.

I. Newton
(1643–1727)

K. Hensel
(1861–1941)



M. Rosenlicht
(1927–1999)

Tutorial on Hardy fields and transseries

Part IV: Model theory of transseries

Matthias Aschenbrenner

universität wien

Kurt Gödel Research Center for Mathematical Logic
Department of Mathematics

Plan for my lectures

I. Hardy fields
II. Transseries
III. Asymptotic differential algebra
IV. Model theory of transseries
V. Maximal Hardy fields

IV. Model theory of transseries

Reminder about yesterday

From the last lecture recall:

An H-*field* is an ordered differential field H such that:

(H1) $f \succ 1 \;\Rightarrow\; f^\dagger > 0$;

(H2) $f \asymp 1 \;\Rightarrow\; f \sim c$ for some $c \in C^\times$;

(H3) $f \prec 1 \;\Rightarrow\; f' \prec 1$.

Examples: ordered differential subfields $\supseteq \mathbb{R}$ of $\mathbb{T}$; Hardy fields $\supseteq \mathbb{R}$.

Let H be an H-field, $f, g \in H^\times$. We have $f^\dagger \succ g'$ if $f, g \prec 1$.

Three flavors of H-fields

- *grounded*: there is an $f \prec 1$ such that $g^\dagger \succcurlyeq f^\dagger$ for all $g \prec 1$;
- *with a gap*: there is a γ with $f^\dagger \succ \gamma \succ g'$ for all $f, g \prec 1$;
- *asymptotic integration*: for each f there is g with $g' \asymp f$.

Reminder about yesterday

We call a real closed H-field *Liouville closed* if all equations

$$y' = f \qquad \text{and} \qquad z^\dagger = g \ (z \neq 0)$$

have solutions. (Examples: $\mathbb{T}$.)

A *Liouville closure* of an H-field H is a minimal Liouville closed H-field extension of H. (Example: $\mathrm{Li}(H)$ for a Hardy field $H \supseteq \mathbb{R}$.)
Every H-field H has one or two Liouville closures.

ω-freeness

A nice property of an ungrounded H-field H that guarantees:

- H has asymptotic integration and a unique Liouville closure;
- the Newton polynomial of each $P \in H\{Y\}^{\neq}$ has a simple shape.

ω-freeness is preserved under d-algebraic H-field extensions:
an H-field extension E of H is **differentially algebraic** (d-algebraic) if for each $y \in E$ there is a $P \in H\{Y\}^{\neq}$ with $P(y) = 0$.

Reminder about yesterday

Let H be an ungrounded H-field and $C = C_H, \Gamma = \Gamma_H$.

Let $P \in H\{Y\}^{\neq}$.

- *Newton polynomial* $N_P \in C\{Y\}^{\neq}$ of P:

  for sufficiently small ("active") ϕ,

$$P^{\phi} \sim \mathfrak{d} \cdot N_P \qquad \text{where } \mathfrak{d} = \mathfrak{d}_{\phi} \in H^{\times};$$

- *Newton degree* of P: $\operatorname{ndeg} P := \deg N_P$.

We say that H is *newtonian* if every $P \in H\{Y\}^{\neq}$ with $\operatorname{ndeg} P = 1$ has a zero $y \preccurlyeq 1$. (Mostly useful in combination with ω-freeness.)

Our first aim: to extend H to a newtonian H-field.

An H-field extension E of H is **immediate** if $C_E = C$ and $\Gamma_E = \Gamma$: for each $g \in E^\times$ there is an $h \in H^\times$ with $g \sim h$.

One can show that H has an immediate H-field extension which is *maximal*, i.e., has no proper immediate H-field extension.

The proof of the next important fact uses the full machinery of Newton diagrams, including its most complicated part (differential Tschirnhaus transformations) to deal with "almost multiple zeros".

Theorem (characterization of newtonianity for ω-free H)

H is newtonian $\iff$ $\left\{ \text{\textit{$H$ has no proper immediate d-algebraic H-field extension.}} \right.$

(We assumed Γ divisible; general version is due to N. Pynn-Coates.)

The previous theorem implies that each H-field can be embedded into a newtonian H-field. We now want to do this in a minimal way.

Definition

A **newtonization** of H is a newtonian extension of H which embeds over H into each newtonian extension of H.

Theorem

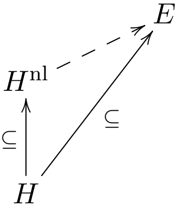
Suppose H is ω-free. Then H has a newtonization. Moreover, if N is a newtonization of H, then

- *N is an immediate extension of H;*
- *no proper ordered differential subfield of N containing H is newtonian.*

We note the following important consequence.

Corollary

If H is ω-free, then H has a newtonian Liouville closed H-field extension H^{nl} which embeds over H into each newtonian Liouville closed H-field extension E of H.



(By alternating newtonization with taking Liouville closures.)

We call H^{nl} the **Newton-Liouville closure** of H.
(Unique up to isomorphism over H.)

The Newton-Liouville closure H^{nl} of H is d-algebraic over H, and its constant field is real closed and algebraic over H.

Definition

Call an H-field **closed** if it is Liouville closed, ω-free, and newtonian.

Thus every H-field extends to a closed one, and $\mathbb{T}$ is closed.

An important fact characterization of closed H-fields:

Theorem ("no new constants")

H *is closed* $\iff$ $\left\{\right.$ C *is real closed and* H *has no proper* d*-algebraic* H*-field extension with constant field* C.

(For a generalization of $\Rightarrow$, and caveats about applying the theorem, see *Relative differential closure in Hardy fields*, `arXiv:2412.10764`.)

Example

The H-subfield $\mathbb{R}(\ell_0, \ell_1, \dots)$ of $\mathbb{T}$ is ω-free. Its Newton-Liouville closure inside $\mathbb{T}$ is $\mathbb{T}^{\mathrm{da}} := \{f \in \mathbb{T} : f$ is d-algebraic over $\mathbb{Q}\}$.

Let $\mathcal{L} = \{0, 1, +, \cdot, \partial, \leqslant, \preccurlyeq\}$ and let

$$T = \text{the } \mathcal{L}\text{-theory of closed } H\text{-fields,}$$

that is, the $\mathcal{L}$-theory axiomatized by

- the axioms for Liouville closed H-fields;
- the ω-freeness axiom; and
- the axiom scheme of newtonianity.

$\mathbb{T}$-Conjecture, revised version

T is model complete.

This can be phrased in terms of sub-H-algebraic sets like the first version of the $\mathbb{T}$-Conjecture. Alternatively: *if H is a closed H-field, then each system of finitely many conditions*

$$P(Y)\,\varrho\,Q(Y) \quad \left\{ \begin{array}{l} \text{where } P, Q \in H\{Y\} = H\{Y_1, \dots, Y_n\} \text{ and } \varrho \\ \text{is one of the symbols } =, \neq, \leqslant, <, \preccurlyeq, \prec, \end{array} \right.$$

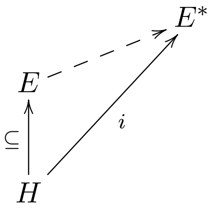
which has a solution in <u>some</u> H-field extension of H, has one in H.

Proof of the $\mathbb{T}$-Conjecture

Theorem (main result of our book)

The refined $\mathbb{T}$-Conjecture is true!

We explain the proof strategy. By a model completeness test of A. Robinson, it suffices to solve the following embedding problem:



Let H be an ω-free H-subfield of some closed E.

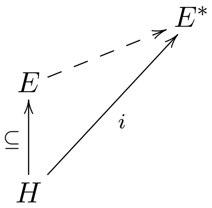
Let i be an embedding of H into a “very rich” closed H-field E^*. Then i extends to an embedding $E \hookrightarrow E^*$.

We first make some preliminary reductions. First, we can extend i to an embedding $H(C_E) \hookrightarrow E^*$. Since $H(C_E)$ is d-algebraic over H, it remains ω-free.

Theorem (main result of our book)

The refined $\mathbb{T}$-Conjecture is true!

We explain the proof strategy. By a model completeness test of A. Robinson, it suffices to solve the following embedding problem:



Let H be an ω-free H-subfield of some closed E such that $C = C_E$.

Let i be an embedding of H into a "very rich" closed H-field E^*. Then i extends to an embedding $E \hookrightarrow E^*$.

Next, suppose there is a $y \in E$ with $C < y < H^{>C}$. Then $H\langle y \rangle$ is grounded, but it extends to an ω-free H-field

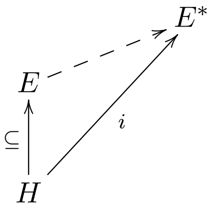
$$\text{“}H\langle y\rangle_\omega = H\langle y, \log y, \log\log y, \dots\rangle\text{”}$$

in a canonical way. Now i extends to an embedding $H\langle y\rangle_\omega \hookrightarrow E^*$.

Theorem (main result of our book)

The refined $\mathbb{T}$-Conjecture is true!

We explain the proof strategy. By a model completeness test of A. Robinson, it suffices to solve the following embedding problem:



Let H be an ω-free H-subfield of some closed E such that $C = C_E$ and $H^{>C}$ is coinitial in $E^{>C}$.

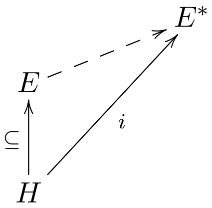
Let i be an embedding of H into a "very rich" closed H-field E^*. Then i extends to an embedding $E \hookrightarrow E^*$.

This has the nice consequence that now we don't need to worry about preserving ω-freeness anymore: *every* differential subfield of E containing H is an ω-free H-subfield of E.

Theorem (main result of our book)

The refined $\mathbb{T}$-Conjecture is true!

We explain the proof strategy. By a model completeness test of A. Robinson, it suffices to solve the following embedding problem:



Let H be an ω-free H-subfield of some closed E such that $C = C_E$ and $H^{>C}$ is coinitial in $E^{>C}$.

Let i be an embedding of H into a "very rich" closed H-field E^*. Then i extends to an embedding $E \hookrightarrow E^*$.

Replacing H by $H^{\mathrm{nl}} \subseteq E$ we can further assume that H is closed. Then each $y \in E \setminus H$ is d-transcendental over H, and the isomorphism type of $H\langle y \rangle$ over H is (essentially) determined by the cut $H^{<y} = \{f \in H : f < y\}$ of y in the ordered set H. $\square$

Corollary

1. *any two closed H-fields are elementarily equivalent; hence*
2. $\mathbb{T}$ *is decidable.*

Proof.

Part 1 is an immediate consequence of the $\mathbb{T}$-Conjecture and the fact that $\mathbb{T}^{\mathrm{da}}$ embeds into every closed H-field. Part 2 follows from 1 and Gödel's Completeness Theorem. □

An instance of 2: *there is an algorithm which, given* d*-polynomials*

$$P_1, \dots, P_m \in \mathbb{Q}(x)\{Y_1, \dots, Y_n\},$$

decides whether $P_1(y) = \cdots = P_m(y) = 0$ *for some* $y \in \mathbb{T}^n$.

(Not true for $\mathbb{T}_{\exp}$!)

Strengthening the $\mathbb{T}$-Conjecture

Recently E. Kaplan established a version of the $\mathbb{T}$-Conjecture when $\mathbb{T}$ is also equipped with $(c, f) \mapsto f^c \colon \mathbb{R} \times \mathbb{T}^{>} \to \mathbb{T}^{>}$.

(But we do not know whether a version of the $\mathbb{T}$-Conjecture holds when $\mathbb{T}$ is expanded to an $\mathcal{L}_{\text{an,exp}}$-structure.)

We remark that we obtained the $\mathbb{T}$-Conjecture in a strengthened form (quantifier elimination in a slight extension of our language $\mathcal{L}$).

Rather than explaining this strengthening, we will discuss a few of its remarkable consequences for $\mathbb{T}$.

Consequences of the $\mathbb{T}$-Conjecture, 2

Corollary

1. *$\mathbb{T}$ is o-minimal at $+\infty$: if $X \subseteq \mathbb{T}$ is sub-H-algebraic, then there is an $f \in \mathbb{T}$ with $(f, +\infty) \subseteq X$ or $(f, +\infty) \cap X = \emptyset$.*
2. *All sub-H-algebraic subsets of $\mathbb{R}^n \subseteq \mathbb{T}^n$ are semialgebraic.*

Special case of 1: if $P \in \mathbb{T}\{Y\}$, then there are $f \in \mathbb{T}$ and $\sigma \in \{\pm 1\}$ with $\operatorname{sign} P(y) = \sigma$ for all $y > f$. (Related to results of Borel, Hardy.)

An illustration of 2: the set of $(c_0, \ldots, c_n) \in \mathbb{R}^{n+1}$ such that

$$c_0 y + c_1 y' + \cdots + c_n y^{(n)} = 0, \qquad 0 \neq y \prec 1$$

has a solution in $\mathbb{T}$ is a semialgebraic subset of $\mathbb{R}^{n+1}$.

We already learned: each Hardy field H extends to the Liouville closed Hardy field $\operatorname{Li}\big(H(\mathbb{R})\big)$. With more work (`arXiv:2404.03695`):

H also extends to an ω-free Hardy field $\supseteq \mathbb{R}$.

Using Newton-Liouville closures (and results from our QE), this yields:

Corollary (extending expansion operators)

Let E be a d-algebraic Hardy field extension of a Hardy field H. Then every embedding $H \to \mathbb{T}$ extends to an embedding $E \to \mathbb{T}$.

As a consequence of this and a theorem of Lion-Miller-Speissegger,

the Hardy field of the Pfaffian closure of the ordered field of real numbers embeds into $\mathbb{T}^{\mathrm{da}}$.

Question

Let $H(\boldsymbol{R})$ be the Hardy field of an o-minimal expansion $\boldsymbol{R}$ of the ordered field of reals with Pfaffian closure $\operatorname{Pf}(\boldsymbol{R})$. Does each embedding $H(\boldsymbol{R}) \to \mathbb{T}$ extend to an embedding $H\big(\operatorname{Pf}(\boldsymbol{R})\big) \to \mathbb{T}$?

It is natural to wonder:

are there Hardy fields $\supseteq \mathbb{R}$ which are closed H-fields?

Definition

A Hardy field is d**-maximal** if it has no proper d-algebraic Hardy field extension.

Maximal Hardy fields (with respect to inclusion) are d-maximal, and d-maximal Hardy fields contain $\mathbb{R}$ (thus are H-fields) and are Liouville closed and ω-free.

Boshernitzan (1986):

each $y \in \mathcal{C}^2$ with $y'' + y = \mathrm{e}^{x^2}$ is hardian, and every d-maximal Hardy field contains a solution y to this equation.

(So there are at least $2^{\aleph_0}$ many maximal Hardy fields; in fact, later it turned out that there are exactly $2^{2^{\aleph_0}}$ many.)

Here is the fundamental fact about d-maximal Hardy fields:

Theorem (characterizing d-maximal Hardy fields)

Let H be a Hardy field. Then

$$H \text{ is } \mathrm{d}\text{-maximal} \quad \Longleftrightarrow \quad H \supseteq \mathbb{R} \text{ and } H \text{ is closed.}$$

Combining it with known properties of closed H-fields substantiates this as the "ultimate" d-algebraic extension theorem for Hardy fields:

Corollary

Let H be a Hardy field and $P \in H\{Y\}$, $P \notin H$.

1. *There are y, z in a Hardy field extending H with $P(y + z\mathrm{i}) = 0$.*
2. *If P has odd degree, then there is some y in a Hardy field extension of H with $P(y) = 0$.*

Closed Hardy fields

For a proof of this theorem see `arXiv:2408.05232`.

THE THEORY OF MAXIMAL HARDY FIELDS

MATTHIAS ASCHENBRENNER, LOU VAN DEN DRIES, AND JORIS VAN DER HOEVEN

Abstract. We show that all maximal Hardy fields are elementarily equivalent as differential fields to the differential field $\mathbb{T}$ of transseries, and give various applications of this result and its proof.

Contents

We give a rough outline in the next lecture, together with some applications and extensions.

In the remainder of this lecture we prepare the ground by talking a bit about *linear differential operators* over H-fields.

Let K be a differential field and $C = C_K$. We put

$$K[\partial] = \text{the ring of linear differential operators over } K.$$

Formally, $K[\partial]$ is a ring containing K as a subring, with a distinguished element ∂, such that as K-vector space,

$$K[\partial] = K \oplus K\partial \oplus K\partial^2 \oplus \cdots$$

and

$$\partial a = a\partial + a' \text{ for all } a \in K.$$

Every $A \in K[\partial]$ can be written as

$$A = a_0 + a_1\partial + \cdots + a_r\partial^r \quad (a_0, \ldots, a_r \in K,\ r \in \mathbb{N}).$$

If $a_r \neq 0$, then A has **order** r, and if $a_r = 1$, then A is called **monic**. With $\operatorname{order}(0) := -\infty$, we have

$$\operatorname{order}(AB) = \operatorname{order}(A) + \operatorname{order}(B) \quad \text{for all } A, B \in K[\partial].$$

Call A of positive order **irreducible** if there are no $A_1, A_2 \in K[\partial]$ of positive order with $A = A_1 A_2$. Each A of positive order factors as

$$A = A_1 \cdots A_n \quad \text{with irreducible } A_1, \dots, A_n \in K[\partial].$$

Let R be a differential ring extension of K. With A as above we obtain a C-linear operator

$$y \mapsto A(y) := a_0 y + a_1 y' + \cdots + a_r y^{(r)} \colon R \to R.$$

Multiplication in $K[\partial]$ $\longleftrightarrow$ composition of C-linear operators:

$$(AB)(y) = A\big(B(y)\big) \quad \text{for } A, B \in K[\partial] \text{ and } y \in R.$$

The kernel of $A \in K[\partial]$ acting as C-linear operator on R,

$$\ker_R A := \big\{ y \in K : A(y) = 0 \big\},$$

is a C-linear subspace of R, with $\dim \ker_K A \leqslant r$ if $0 \leqslant \operatorname{order} A \leqslant r$.

The following are the main results about linear differential operators over H-fields. *For this let H be a closed H-field and $K := H[i]$, equipped with the unique derivation extending that of H.*

Theorem (factorization of operators)

Every irreducible element of $K[\partial]$ has order 1. As a consequence, if $A \in H[\partial]$ is irreducible, then A has order 1 or order 2.

Proof.

Let $A \in K[\partial]$ have order $r \geqslant 1$. There exists $R \in K\{Z\} \setminus K$ (the *Riccati transform* of A) of order $r-1$ such that

$$\frac{A(y)}{y} = R(z) \qquad \text{for each unit } y \text{ of a } \mathrm{d}\text{-ring} \supseteq K \text{ and } z = y^{\dagger}.$$

Since H is closed, we can take some $z \in K$ with $R(z) = 0$; then we have $A = B \cdot (\partial - z)$ for some $B \in K[\partial]$. □

Hence in order to establish the newtonianity of d-maximal Hardy fields, we will have to, in particular, deal with factoring linear differential operators over complexified Hardy fields $K = H[\mathrm{i}]$.

This actually turns out to be a key tool for proving the characterization of d-maximal Hardy fields.

The final fact for today uses only that K, equipped with the dominance relation extending that of H (a d-valued field in the sense of Rosenlicht) is newtonian, in the natural sense:

Theorem (linear surjectivity)

Let $A \in K[\partial]^{\neq}$. Then for each $b \in K$ there is a $y \in K$ with $A(y) = b$.

E. Borel
(1871–1956)

A. Robinson
(1918–1974)

M. Boshernitzan
(1950–2019)

Tutorial on Hardy fields and transseries

Part V: Maximal Hardy fields

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V. Maximal Hardy fields

Recall that

- an H-field is *closed* if it is Liouville closed, ω-free, newtonian;
- a Hardy field is d-*maximal* if it has no proper d-algebraic Hardy field extension.

At the end of yesterday's lecture we met the fundamental

Characterization of d-maximality

Let H be a Hardy field. Then

$$H \text{ is } \mathrm{d}\text{-maximal} \quad \Longleftrightarrow \quad H \supseteq \mathbb{R} \text{ and } H \text{ is closed.}$$

$\Leftarrow$ Follows from the "no new constants" theorem, also discussed yesterday.

$\Rightarrow$ Amounts to showing that each Hardy field has a d-algebraic Hardy field extension $H \supseteq \mathbb{R}$ which is closed.

Today we tackle this remaining task:

Constructing newtonian Hardy fields

Theorem

Every ω-free Hardy field has a newtonian Hardy field extension.

Recall: an ungrounded H-field H is *newtonian* if every $P \in H\{Y\}^{\neq}$ with $\operatorname{ndeg} P = 1$ has a zero $y \preccurlyeq 1$ in H.

This notion, and ω-freeness, also make sense (and have equally nice properties) for d-*valued fields* such as $K = H[\mathrm{i}]$ for an H-field H:

Definition (☡ nonstandard!)

A valued d-field K with $C = C_K$ is d**-valued** if for $f, g \in K^{\times}$:

(H1′) $f \preccurlyeq g \prec 1 \quad \Rightarrow \quad g' \neq 0 \ \& \ \frac{f}{g} - \frac{f'}{g'} \prec 1 \ \& \ f^{\dagger} \succcurlyeq g^{\dagger}$;

(H2) $f \asymp 1 \quad \Rightarrow \quad f \sim c$ for some $c \in C^{\times}$;

(H3) $f \prec 1 \quad \Rightarrow \quad f' \prec 1$.

Let K be an ω-free d-valued field.

Definition

A **hole** in K is a triple $(P, \mathfrak{m}, \widehat{f})$ where

- $P \in K\{Y\} \setminus K$,
- $\mathfrak{m} \in K^\times$, and
- $\widehat{f} \in \widehat{K} \setminus K$ for an immediate d-valued field extension $\widehat{K}$ of K,

such that $P(\widehat{f}) = 0$ and $\widehat{f} \prec \mathfrak{m}$.

The **order** and **complexity** of a hole $(P, \mathfrak{m}, \widehat{f})$ in K are those of P.
A hole in K is **minimal** if no hole in K has smaller complexity; these are "minimal counterexamples" to non-newtonianity, since:

K newtonian $\iff$ K has no holes.

(Stated for ω-free H-fields in the last lecture.)

Let $H \supseteq \mathbb{R}$ be an ω-free Liouville closed Hardy field which is <u>not</u> newtonian. To show:

> H has a proper d-algebraic Hardy field extension.

Take a minimal hole $(P, \mathfrak{m}, \widehat{f})$ in H, and arrange $\mathfrak{m} = 1$. Then

- $r := \operatorname{order} P \geqslant 1$;
- $P \in H\{Y\} \setminus H$ is a *minimal annihilator* of $\widehat{f}$ over H
  (i.e., of minimal complexity such that $P(\widehat{f}) = 0$);
- H is $(r-1)$-*newtonian*: "newtonian up to order $r-1$".

> We try to find f in a Hardy field $\supseteq H$ and a ordered differential field isomorphism $H\langle f \rangle \xrightarrow{\cong} H\langle \widehat{f} \rangle$ over H with $f \mapsto \widehat{f}$.

So at the very least:

> We need some $f \in \mathcal{C}^{<\infty}$ with $P(f) = 0$ and $f \prec 1$!

Let $\boxed{S_P := \dfrac{\partial P}{\partial Y^{(r)}}}$ (the **separant** of P), so $\deg_{Y^{(r)}} S_P < \deg_{Y^{(r)}} P$.

Proposition (automatic smoothness)

Let $f \in \mathcal{C}^r$ (so $P(f) \in \mathcal{C}$ makes sense). If $P(f) = 0$ and $S_P(f) \in \mathcal{C}^\times$, then $f \in \mathcal{C}^{<\infty}$.

(Similarly with $\mathcal{C}^\infty$ or $\mathcal{C}^\omega$ in place of $\mathcal{C}^{<\infty}$, provided H is a $\mathcal{C}^\infty$-Hardy field or a $\mathcal{C}^\omega$-Hardy field.)

Relevant case: "almost linear" P

Suppose

$$P = Q + R \qquad \text{where } Q = Y^{(r)} + g_1 Y^{(r-1)} + \cdots + g_0 Y \text{ and } R \prec 1.$$

If $f \in \mathcal{C}^r$ with $P(f) = 0$ and $f, f', \ldots, f^{(r)} \preccurlyeq 1$, then $f \in \mathcal{C}^{<\infty}$.

(Since then $S_P(f) \sim 1$ and so $S_P(f) \in \mathcal{C}^\times$.)

Normalization

We consider various operations on holes in H, such as

> *replace* $(P, \mathfrak{m}, \widehat{f})$ *by* $(P_{+f}, \mathfrak{m}, \widehat{f} - f)$ *where* $f \in H$ *satisfies* $\widehat{f} - f \prec \mathfrak{m}$, *and* $P_{+f}(Y) := P(Y + f)$,

to transform our given hole in H into a hole $(P, 1, \widehat{f})$ in H where P has a nice shape as in the "relevant case" above (and more).

Normalization procedures of this kind are the subject of our monograph `arXiv:2403.19732`.

A NORMALIZATION THEOREM IN ASYMPTOTIC DIFFERENTIAL ALGEBRA

MATTHIAS ASCHENBRENNER, LOU VAN DEN DRIES, AND JORIS VAN DER HOEVEN

Abstract. We define the universal exponential extension of an algebraically closed differential field and investigate its properties in the presence of a nice valuation and in connection with linear differential equations. Next we prove normalization theorems for algebraic differential equations over H-fields, as a tool in solving such equations in suitable extensions. The results in this monograph are essential in our work on Hardy fields in [6].

Contents

Let $$L = L_P := \sum_{i=0}^{r} \left(\frac{\partial P}{\partial Y^{(i)}}\right)(0)\,\partial^i \in H[\partial]$$ (the **linear part** of P).

Can also arrange that $\operatorname{order} L = r$ and L is monic. We now make a

Bold assumption: L splits (strongly) over H

$L = (\partial - \phi_1)\cdots(\partial - \phi_r)$ for some $\phi_j \in H$ with $\phi_j \succcurlyeq 1$.

For a suitable $a \in \mathbb{R}$, representing the coefficients of P by functions in $\mathcal{C}_a$ we obtain an $\mathbb{R}$-linear operator $y \mapsto L(y) \colon \mathcal{C}_a^r \to \mathcal{C}_a$. Here

$$\mathcal{C}_a^r := \left\{ \begin{array}{l} \mathbb{R}\text{-linear space of } r\text{-times continuously differen-} \\ \text{tiable functions } [a,+\infty) \to \mathbb{R}. \end{array} \right.$$

Using our strong splitting of L, by suitable r-fold integrations we obtain a "good" $\mathbb{R}$-linear right inverse $L^{-1} \colon \mathcal{C}_a \to \mathcal{C}_a^r$ of this operator:

$$L\big(L^{-1}(y)\big) = y \qquad \text{for all } y \in \mathcal{C}_a.$$

More precisely, consider the $\mathbb{R}$-linear subspace

$$(\mathcal{C}_a^r)^{\preccurlyeq} := \{f \in \mathcal{C}_a^r : \|f\|, \|f'\|, \ldots, \|f^{(r)}\| < \infty\}$$

of $\mathcal{C}_a^r$, where $\|\cdot\|$ is the sup norm on $\mathcal{C}_a$. Equipped with the norm

$$f \mapsto \|f\|_r := \max\{\|f\|, \|f'\|, \ldots, \|f^{(r)}\|\},$$

this is a Banach space, and “good” roughly means that $L^{-1}: \mathcal{C}_a \to \mathcal{C}_a^r$ restricts to a continuous operator $(\mathcal{C}_a)^{\preccurlyeq} \to (\mathcal{C}_a^r)^{\preccurlyeq}$.

Next we convert the problem of solving

$$P(y) = 0, \qquad y \prec 1$$

in $(\mathcal{C}_a^r)^{\preccurlyeq}$ into a fixed point problem:

Computing a fixed point

Write

$$P = P_1 - R \qquad \text{where } P_1 := \text{homogeneous part of degree } 1 \text{ of } P$$

and consider the (generally non-linear) operator

$$f \mapsto \Psi(f) := L^{-1}\big(R(f)\big) : \mathcal{C}_a^r \to \mathcal{C}_a^r.$$

Then

$$\Psi(f) = f \quad \Longrightarrow \quad R(f) = L(f) = P_1(f) \quad \Longrightarrow \quad P(f) = 0.$$

Now <u>if</u>

- $\Psi(f) \prec 1$ for each $f \in (\mathcal{C}_a^r)^{\preccurlyeq}$, and
- Ψ restricts to a contractive operator on a closed ball of $(\mathcal{C}_a^r)^{\preccurlyeq}$, say $B := \big\{f \in \mathcal{C}_a^r : \|f\|_r \leqslant 1/2\big\}$,

<u>then</u> we get a fixed point $f \in B$ of Ψ with $f \prec 1$ as required.

These hypotheses can be achieved with the right kind of normalization theorem and suitably modifying the definition of Ψ.

Passing to the complex realm

There is a problem with our “bold assumption”:

L might not split over H, or even over $K = H[i]$.

To get around this, we use: *K is also ω-free and non-newtonian.*

So instead of a hole of minimal complexity in H, we let $(P, \mathfrak{m}, \widehat{f})$ be a hole of minimal complexity <u>in K</u>.

We can arrange here that $\widehat{f} = \widehat{g} + \widehat{h}i$ with $\widehat{g}, \widehat{h} \prec 1$ in an immediate H-field extension of H. Then $\widehat{g} \notin H$ or $\widehat{h} \notin H$, say $\boxed{\widehat{g} \notin H.}$

As before we get $r := \operatorname{order} P \geqslant 1$, P is a minimal annihilator of $\widehat{f}$ over K, and K is $(r-1)$-newtonian. We also arrange that $\mathfrak{m} = 1$ and the linear part $L_P \in K[\partial]$ of P has order r. Now indeed:

L_P splits over K.

(Since K is algebraically closed, ω-free, and $(r-1)$-newtonian.)

Our fixed point construction then adapts to produce a germ

$$f = g + hi \quad (g, h \in \mathcal{C}^{<\infty}) \qquad \text{with} \qquad P(f) = 0,\ f \prec 1.$$

Let Q be a minimal annihilator of $\widehat{g}$ over H. We face a new problem:

We cannot expect that $Q(g) = 0$.

If $L_Q \in H[\partial]$ splits over K, then we can try to apply fixed point arguments like the ones above, with $(P, 1, \widehat{f})$ replaced by the hole $(Q, 1, \widehat{g})$ in H, to find a zero $y \in \mathcal{C}^{<\infty}$ of Q.

Unfortunately we only know that $1 \leqslant s \leqslant 2r$ for $s := \operatorname{order} Q$, and we may have $s > r$.

☹ So we cannot ensure that L_Q splits over K, or to normalize $(Q, 1, \widehat{g})$ as we indicated above for $(P, 1, \widehat{f})$.

Consider $L_{Q_{+\widehat{g}}} \in \widehat{H}[\partial]$ (which also has order s).

A differential-algebraic fact to the rescue:

If $L_{P_{+\widehat{f}}}$ splits over $\widehat{K}$, then so does $L_{Q_{+\widehat{g}}}$.

The hypothesis here holds if H is *dense* in $\widehat{H}$ (in the sense of $\preccurlyeq$):

for all $\widehat{y} \in \widehat{H}$ and $\varepsilon \in H^{\times}$ there is a $y \in H$ with $y - \widehat{y} \prec \varepsilon$.

In fact, if $g \in H$ is only sufficiently close to $\widehat{g}$, then $L_{Q_{+g}} \in H[\partial]$ is close to an operator in $H[\partial]$ that does split over K.

Idea

Use $(Q_{+g}, 1, \widehat{g} - g)$ instead of $(Q, 1, \widehat{g})$.

This *almost* works, but:

We can neither expect that H is dense in $\widehat{H}$, nor that the hole $(Q, 1, \widehat{g})$ in H is minimal.

Fortunately, to get around this we can instead

- use that $\widehat{g}$ is the limit of an “almost” cauchy sequence in H;
- in the definition of “hole” in H relax the condition $Q(\widehat{g}) = 0$.

Now suppose we finally find $g \in \mathcal{C}^{<\infty}$ such that $Q(g) = 0$ and $g \prec 1$.

We need to adjoin g to H:

To show

The germ g generates a Hardy field $H\langle g\rangle$ isomorphic to $H\langle\widehat{g}\rangle$ by an isomorphism over H with $g \mapsto \widehat{g}$.

The zeros $g, \widehat{g}$ of Q must have similar asymptotic properties w. r. t. H:

Example (we need to show much more, of course)

$$h, \mathfrak{n} \in H \quad \& \quad \widehat{g} - h \prec \mathfrak{n} \qquad \Longrightarrow \qquad g - h \prec \mathfrak{n}.$$

Now $(g-h)/\mathfrak{n}$ and $(\widehat{g}-h)/\mathfrak{n} \prec 1$ are zeros of $Q_{+h,\times\mathfrak{n}} \in H\{Y\}$.
The Fixed Point Theorem also yields a zero $y \prec 1$ of $Q_{+h,\times\mathfrak{n}}$ in $\mathcal{C}^{<\infty}$.
Then $\boxed{g}$ and $\boxed{g_1 := y\mathfrak{n} + h}$ both solve the asymptotic equation

$$Q(Y) \;=\; 0, \qquad Y \prec 1. \tag{E}$$

If we can get $\boxed{g - g_1 \prec \mathfrak{n}}$ then $g - h = (g - g_1) + y\mathfrak{n} \prec \mathfrak{n}$ as needed.

Call a germ $\phi \in \mathcal{C}$ *small* if $\phi \prec \mathfrak{n}$ for all $\mathfrak{n} \in H^{\times}$ with $\widehat{g} - h \prec \mathfrak{n}$ for some $h \in H$. Thus we need to show:

differences of solutions to (E) *in* $\mathcal{C}^{<\infty}$ *are small.*

Simple estimates coming out of the proof of the Fixed Point Theorem are not enough. We need

- a generalization of the Fixed Point Theorem for *weighted norms* (instead of $\|\cdot\|_r$) with weight given by a representative of $\mathfrak{n}$;
- a construction of right-inverses of linear differential operators which is "uniform in $\mathfrak{n}$".

We use this to show:

each difference ϕ *of solutions to* (E) *give rise to a zero* $z \prec 1$ *of* A *in* $\mathcal{C}^{<\infty}[\mathrm{i}]$ *whose smallness implies that of* ϕ.

Here $A \in H[\partial]$ is a linear differential operator of order s which approximates L_Q and splits over K, implicit in the above.

To ensure that all zeros of this operator A are small requires another normalization procedure on $(Q, 1, \widehat{g})$.

To make all this work, we also need to study the asymptotics of zeros of linear differential operators which split over K.

For this we rely on a theorem of Boshernitzan on uniform distribution mod 1 of hardian germs, combined with a structure theorem for the kernel $\boxed{\ker_{\mathcal{C}^{<\infty}[i]} A}$ of A. □

Rather than going into more detail, we now conclude this sketch of the characterization of d-maximality, and just formulate the relevant structure theorem, in the case of matrix linear differential equations over d-maximal H:

Solution spaces of linear differential equations

Generalizing a fundamental theorem about holonomic functions:

Theorem (assuming H is d-maximal)

Let M be an $n \times n$ matrix over $K := H[\mathrm{i}]$. Then the $\mathbb{C}$-linear space of solutions (in $\mathcal{C}^{<\infty}[\mathrm{i}]$) to the linear differential equation $\boxed{y' = My}$ *has a basis*

$$f_1\,\mathrm{e}^{\phi_1 \mathrm{i}}, \ldots, f_n\,\mathrm{e}^{\phi_n \mathrm{i}} \qquad \text{where } f_j \in K^n,\ \phi_j \in H\ (j = 1, \ldots, n).$$

The $\phi_j' \mathrm{i}$ are “eigenvalues” of $y' = My$.

Can arrange here $\phi_j = 0$ or $\phi_j \succ 1$, and $\phi_i = \phi_j$ or $\phi_i - \phi_j \succ 1$.

If M has suitable symmetries, then we can also guarantee the existence of a nonzero solution which lies in K^n (and hence is non-oscillatory): e.g., if M *is skew-symmetric* and n *is odd*.

Corollary (conjectured by Boshernitzan, 1982)

Let H be a Hardy field and $a, b \in H$, and suppose the equation

$$y'' + ay' + by = 0 \tag{L}$$

has an oscillating solution (in $\mathcal{C}^2$). Then there are germs $g > 0$ and $\phi > \mathbb{R}$ in a Hardy field extension of H such that

$$y \text{ is a solution of (L)} \iff y = cg\cos(\phi + d) \text{ for some } c, d \in \mathbb{R}.$$

In favorable situations (e.g., when H is ω-free), g, ϕ are unique up constants, and contained in each maximal Hardy field containing H.

The example of the Bessel equation

$$x^2Y'' + xY' + (x^2 - \nu^2)Y = 0 \qquad (\mathsf{B}_\nu)$$

Corollary

There is a unique germ $\phi = \phi_\nu$ in some Hardy field with $\phi - x \preccurlyeq 1/x$ such that the solutions of (B_ν) are exactly the germs of the form

$$y = \frac{c}{\sqrt{x\phi'}} \cos(\phi + d) \qquad (c, d \in \mathbb{R}).$$

The “phase function” ϕ_ν is Liouvillian $\Leftrightarrow \nu \in \frac{1}{2} + \mathbb{Z}$, and then

$$\phi = x + \sum_{j=1}^{m} \arctan\left(\tfrac{a_j}{x-b_j}\right) \qquad \text{for distinct pairs } (a_j, b_j) \in \mathbb{R}^\times \times \mathbb{R}.$$

The example of the Bessel equation

We have an asymptotic expansion

$$\phi_\nu \sim x + \frac{\mu-1}{8}x^{-1} + \frac{\mu^2-26\mu+25}{384}x^{-3} + \frac{\mu^3-115\mu^2+1187\mu-1073}{5120}x^{-5} + \cdots$$

with $\mu = 4\nu^2$.

We obtain this by verifying that $\psi = 1/\phi'$ satisfies an order 3 linear differential equation

$$\psi''' + f\psi' + (f'/2)\psi = 0, \quad \psi \sim 1, \qquad f := 4 + (1-\mu)x^{-2}.$$

There is a unique solution to this equation-with-asymptotic-side-condition in $\mathbb{T}$, which can be easily computed explicitly, namely the one on the right-hand side of the asymptotic expansion for ϕ_ν above.

Now embed the Hardy field $\mathbb{R}\langle x, \phi\rangle$ into $\mathbb{T}$ over $\mathbb{R}(x)$ using the expansion theorem from the last lecture. $\square$

(This can be used to prove facts about the Bessel functions—certain distinguished solutions to (B$_\nu$)—in a complex-analysis-free way.)

Dependence on constant coefficients

Many properties of the solutions of $A(y) = 0$ are typically *definable* in the coefficients of $A \in H[\partial]$. For example, let

$$b_1, \ldots, b_r \in H[Z] = H[Z_1, \ldots, Z_m].$$

For $c \in \mathbb{R}^m$ we then obtain a linear differential equation over $H(\mathbb{R})$:

$$y^{(r)} + b_1(c)y^{(r-1)} + \cdots + b_r(c)y = 0 \qquad (\mathrm{L}_c)$$

Corollary (using a result from last lecture)

The set of all $c \in \mathbb{R}^m$ such that no solution $y \in \mathcal{C}^r$ of (L_c) *oscillates is semialgebraic.*

Cauchy-Euler equation, $H = \mathbb{R}(x)$

$y'' + cx^{-1}y' + dx^{-2}y = 0$ ($c, d \in \mathbb{R}$) has no nonzero oscillating solution $\iff (c-1)^2 \geqslant 4d.$

Reminder

A $\mathcal{C}^\omega$*-Hardy field* (also called an *analytic Hardy field*) is a Hardy field $H \subseteq \mathcal{C}^\omega$. (These are the ones of interest for most applications.)

Let M be a *maximal* $\mathcal{C}^\omega$-Hardy field, i.e., a $\mathcal{C}^\omega$-Hardy field which is maximal with respect to inclusion among $\mathcal{C}^\omega$-Hardy fields.

By our characterization of d-maximality and automatic smoothness, M contains $\mathbb{R}$ and is a closed H-field.

Hence each system of finitely many conditions

$$P(Y)\,\varrho\,Q(Y) \qquad \left\{ \begin{array}{l} \text{where } P, Q \in M\{Y\} = M\{Y_1, \dots, Y_n\} \text{ and } \varrho \\ \text{is one of the symbols } =, \neq, \leqslant, <, \preccurlyeq, \prec, \end{array} \right.$$

which has a solution in a Hardy field extension of M, has one in M.

(Likewise with $\mathcal{C}^\infty$ in place of $\mathcal{C}^\omega$.)

We do not know whether M is also a maximal Hardy field!

Nevertheless:

M is dense in each of its Hardy field extensions.

Like the results to follow, this ultimately relies on Whitney's Approximation Theorem.

Maximal $\mathcal{C}^{\omega}$-Hardy fields are very rich. To make this precise, we call an ordered set *short* if it contains no uncountable subset which is well-ordered or reverse well-ordered. (Example: $\mathbb{R}$.)

Theorem (A.-van den Dries, 2025+)

Every short H-field with archimedean constant field embeds into M.

In particular, every Hardy field which is countably generated over its constant field embeds into M.

We already know: the H-field $\mathbb{T}^{\mathrm{da}}$ of differentially algebraic transseries embeds into M. But since $\mathbb{T}$ is short, M even supports a "summation operator" defined on all of $\mathbb{T}$:

Corollary

$\mathbb{T}$ embeds into M.

This is a Hardy field version of Besicovitch's analytic strengthening of Borel's theorem on C^∞-functions with prescribed Taylor series.

The key to the results above is an understanding of singly generated (d-transcendental) Hardy field extensions $H\langle y\rangle \supseteq H$ where y is *of countable type over* H:

the cofinality of $H^{<y}$ and coinitiality of $H^{>y}$ are countable.

This extends earlier results of Boshernitzan and Sjödin mentioned in the first lecture (the case $y > H$). The proof is supported by a careful analysis of singly-generated extensions of asymptotic couples.

We finish with some open

Questions

1. Is there a maximal Hardy field which is closed under composition? Under compositional inversion? [Boshernitzan]
2. Is there a Hardy field H which is closed under composition and an embedding $\mathbb{T} \to H$ which respects composition?

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Merci de votre attention!